GEOMETRY OF THE QUADRANGLE

by

ELLA JANE EATON

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Approved: U.G. Mitchell

Department of Mathematics.

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BIBLIOGRAPHY FOR THE HISTORICAL SKETCH

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It is the object of this paper to give as many of the properties of the complete quadrangle with their history as can be found in the library of the University of Kansas.

The history of the properties of the simple quadrangle has been treated in the chronological order, while the theorems have been classified as follows:-

A. Simple quadrangle

- I. Parallelogram.
  - a. Rectangular.
    - (I) Square.
    - (2) Oblong.
  - b. Nonrectangular.
    - (I) Rhombus.
    - (2) Rhomboid.
- II. Trapeziod.
  - a. Isosceles trapezoid.
  - b. Scalene trapezoid.
- III. Trapezium.

HISTORICAL DEVELOPMENT

Several examples of the simple quad rangle present themselves in so striking a form, that they could hardly have failed to impress the earliest races of mankind. For corroborative evidence of their influence we have only to refer to the early art of such peoples as the Greeks, Celts and Egyptians.

The Geometric Age in Greek Art is so called on account of the use of geometric figures in the decoration of pottery and bronze work as found in the graves of the period from the ninth to the seventh century B.C. In these designs the use of the square and the parallelogram is conspicuous. In the famous Celtic burial grounds in Hallstatt in upper Austria were found shields, plates and hemlets with designs resembling those on the pottery in the graves of Greece. The latest possible date, given by authorities for their origin is 1350 B.C. Examples of this kind might be multiplied. These are sufficient to show that since the simpler figures were familiar, it is not improbable that some of their properties were known at a very early date.

A record of the discovery of these properties is of necessity fragmentary. The history of some of them can be given with certainty while the traditions of others are interesting if not reliable.

According to the Archaeologist, Halprecht the Babylonians, in the second or third milennium B.C., were familiar with the correct methods for finding the areas of the square, rectangle and trapeziod. This information he obtained from examining more or less carefully fifty thousand cuneiform tablets, most of which were taken from the excavations at Nippur.

Our second source of information on ancient mathematics is the Egyptian manuscript written about 2000B.C. This contains no theorems, but special problems, for example;- to measure a rectangle two units by ten units; to describe a trapezoid whose parallel sides are six and four units and each of the other sides twenty units. We find in it also, the incorrect formula for the area of an isosceles trapezoid $\frac{a}{2}(g_I \text{ plus } g_2)$ where a is one of the equal sides and g_I and g_2 are the bases.

Egypt is considered as the birthplace of geometry but to what extent the Egyptians developed it is hard to determine. Democritus and Plutarch speak of their ability to compute areas and their skill in Geometric construction. In connection with the quadrangle this question is of some importance. The Greeks until the time of Archimedes were not interested in mensuration. Most of the early theorems which we have along this line were formulated by men who were acquainted with Egyptian learning, or indirectly through Pythagores had become interested in this subject.

An inscription on the temple of Horns at Edfu in upper Egypt. In a rule for the working out of the priest's estates, the area of an irregular quadrangle was used as $\frac{a+b}{2} \cdot \frac{c+d}{2}$ Where a,b,c,d, were the sides of the quadrangle. It is not as close an approximation as the one given in the Rhuid Papyrus for the isosceles trapezoid. This would lead us to think that the Egyptian Geometry made little advance from the time of Ahmes to the first century B.C. On the other hand we know that such men as Thles and Pythagoras would hardly have spent years of study in Egypt on a few unformulated theorems some of which were incorrect.

It is through Pythagoras that the Geometry of the quadrangle was carried from Egypt into Greece. Thales who must have been as familiar with this side of Egyptian Geometry as his successor did not seem to have developed or to have transferred to his pupils his knowledge.

The work of Pythagoras can not be separated from that of the secret society which he founded. There are several reasons for this. Members were forbidden to publish their work and their discoveries were ascribed to the founder. Prochus tells us that all the theorems concerning the application of areas in the sense used in Book I, Theorem 44 of Euclid, and with the broader meaning as in the VI Book of Euclid, "are old and the inventions of the Pythagorians". This is to some extent confirmed by Plutarch who said that Pythagoras contrary to his religion,"sacrificed an ox on account of the geometric diagram either the one referring to the square on the hypotenuse or that relating to the problem concerning the application of areas." He attributes to Pythagoras the solution of the problem, to construct a rectilineal figure similar to one figure and equivalent to another.

The school founded by Pythagores existed for several centuries. It was from members of this brotherhood that Hippocrates of Chios acquired his knowledge of mathematics. He is the first, of whom we have any account, to break the rule of silence of this order, by publishing in 430 B.C. a text book on elementary geometry. Simplicius preserved some of this work of Hippocrates in his extract from Endemius', History of Geometry. In it, solutions of the following problems were required:- To construct a square which shall be equal to a given rectilinear figure; Tried a line the square on which shall be equal to three times the square on a given line; given two straight lines, construct a trapezoid, such that one of the parallel sides shall be equal to the greater of two given lines and each of the three remaining sides equal to the less. About this trapezoid describe a circle.

About this time considerable confusion prevailed concerning the terms used for the different types of the quadrangle. Euclid in the third century B.C. attempted to classify the various forms. Proclus says that Euclid was the man who introduced the word parallelogram. His use of the word is the same as purs. Under it he included the square, rectangle, rhombus and rhomboid. Euclid did not attempt to add any new properties, of these figures, in his elements. One of the most simple of the properties, the diagonals of a parallellogram bisect each other, was omitted. This first appears in the writings of Archimeded who gave the theorem, the point of intersection of the lines joining the mid points of the opposite sides of a parallelogram coincides with the point of intersection of the diagonals.

Heron in IOO B.C. gave a new classification, which differed in several respects from that of Euclid. Euclid had used the word trapezoid to include all quadrangles which were not parallelograms. Heron distinguishes between parallel trapezoid and irregular quadrangle. He also gave the solution to the problem, to construct two rectangles in which the sum of their sides as well as their areas are in a given ratio.

Passing on to the second Alexandrian School, Ptolmey in about I50 A.D. proved that a rectangle contained by the diagonals of a quadrangle inscribed in a circle is equivalent to both the rectangles contained by its opposite sides. Latter in Pappus', Collection of Mathematics, appears a solution of the problem to find a parallelogram whose sides are in a fixed ratio to those of a given parallelogram while the areas are in another ratio.

This concludes what could be found on the Early History of the Quadrangle, in Greek or Egyptian Mathematics. In India, Brahmagupta and Baschara were acquainted with some of these properties, but they added nothing to the knowledge possessed in the Occident.

SECTION I

The properties of the parallelogram are numerous. Many of them can be found in the text books on elementary geometry. The proofs are usually simple, for such, only the statement of the theorem needs to be given.

The more general theorems will be considered first, followed by those on the special forms of the parallelogram according to the general outline. By a parallelogram is meant of course a quadrangle in which the opposite sides are equal and parallel.

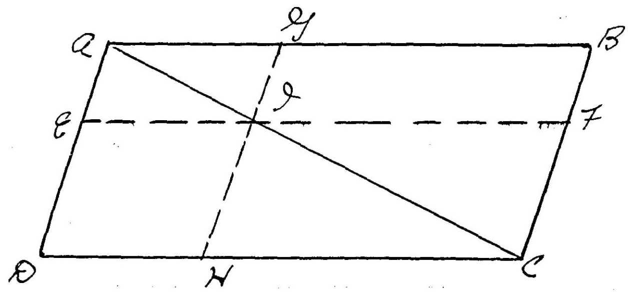
Theorem I. In a parallelogram the diagonals bisect each other.

Theorem II. In a parallelogram a diagonal bisects the area.

Theorem III. Parallelograms which have the same base or equal bases and equal altitudes are equal.

Theorem IV. If a parallelogram has the same base and altitude as a triangle the area of the parallelogram is double that of the triangle.

Theorem V. In any parallelogram the complements of the parallelograms about the diagonals are equal.



Given the parallelogram ABCB, with the parallelograms EG and FH on the diagonal AC.

To prove EH = GF.

Proof: $\triangle$ ADC = $\triangle$ ABC (I)

$\triangle$ AEI = $\triangle$ AGI (2)

$\triangle$ HIC = $\triangle$ IFC (3)

Adding (2) and (3) and subtracting from (I), the parallelogram EH = the parallelogram GF.

Q.E.D.

Theorem VI. In a parallelogram the opposite angles are equal and the adjacent angles are supplementary.

Theorem VII. Two parallelograms are congruent when two adjacent sides and the included angle of one equal respectively to the two adjacent sides and the included angle of the other.

Theorem VIII. Any straight line passing through the diagonals and terminated by the sides bisects the parallelogram.

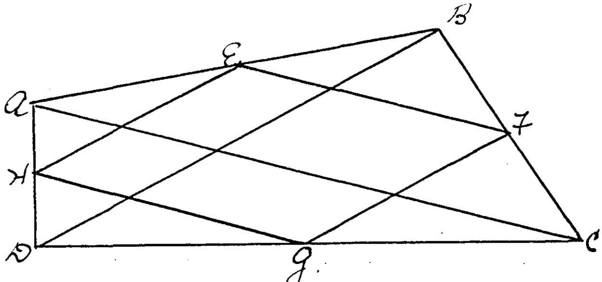
Theorem IX. The areas of the four triangles into which a parallelogram is divided by the diagonals, are equivalent.

Theorem X. If ABCD is a parallelogram, E and F the mid points of AD and BC respectively, BE and DF will bisect the diagonal AC.

Theorem XI. Let ABCD be a parallelogram, P any point in the diagonal BD. If the lines Pa and PC are drawn, triangle PAB will equal triangle PCB. Since they have the same base and equal altitudes.

Theorem XII.

The four lines joining the middle points of the adjacent sides of a quadrangle form a parallelogram.



Let ABCD be one quadrangle, E,F,G,H the midpoints of the sides Ab, BC,CD, DA respectively.

To prove ABCD is a parallelogram.

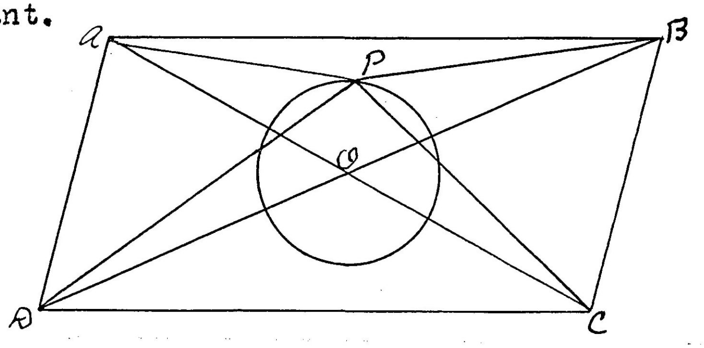
Proof: In the ABD, HE joins the mid points of AD and BE and is therefore parallel to BD. In the same manner FG is parallel to BD.

∴ Bd and HE are parallel.

In the triangle ABC, Ef is parallel to AC, and in ADC, HG is parallel to AC. ∴ EF and HG are parallel. And Ah, Fg is a parallelogram.

Q.E.D.

Theorem XIII. If a circle is described about the point of intersection of the diagonals of a parallelogram as centerm the sum of the squares on the lines drawn from any point on the circumference to the four vertices of the parallelogram is a constant.



Given in the parallelogram ABCD, the circle metr center O the intersection of the diagonals, and P any point on the circumference of this circle.

To prove $\overline{AP}^2 + \overline{PC}^2 + \overline{DP}^2 + \overline{PB}^2 = K$

Proof: $\overline{DP}^2 + \overline{PB}^2 = 2\overline{PO}^2 + 2\overline{DO}^2$

$\overline{AP}^2 + \overline{PC}^2 = 2\overline{PO}^2 + 2\overline{AO}^2$

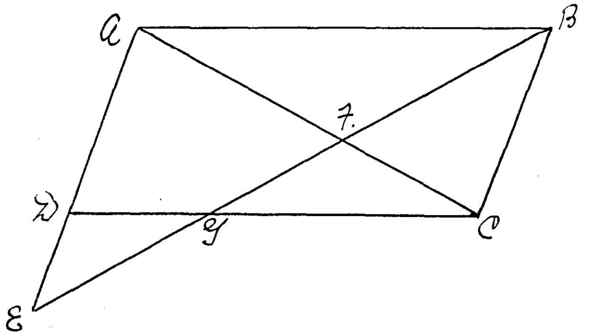
$\therefore \overline{DP}^2 + \overline{PB}^2 + \overline{AP}^2 + \overline{PC}^2 = \overline{4R}^2 + \frac{\overline{BD}}{2}^2 + \frac{\overline{AC}}{2}^2$

But $\overline{4R}^2 + \frac{\overline{BD}}{2}^2 + \overline{AC}^2 = K$

$\therefore \overline{DP}^2 + \overline{PB}^2 + \overline{AP}^2 + \overline{PC}^2 = K$

Q.E.D.

Theorem XIV. If ABCD is a parallelogram from B a line is drawn cutting the diagonal AC at F, the side DC at G, AD produced at E, to prove that $EF \cdot FG = \overline{BF}^2$



To prove: $EF \cdot FG = \overline{BF}^2$

Proof $\quad \frac{EG}{GB} = \frac{DG}{GC}$ (I) $\quad \frac{GF}{GC} = \frac{BF}{AB}$ (2)

$$\frac{DG}{AB} = \frac{EG}{EB} = \frac{EG}{EF + FG} \quad (3)$$

From (I) and (2) $\quad \frac{EG}{GF + FB} = \frac{DG \cdot BF}{GF \cdot AB} = \frac{EG \cdot BF}{GF(EF + FB)}$

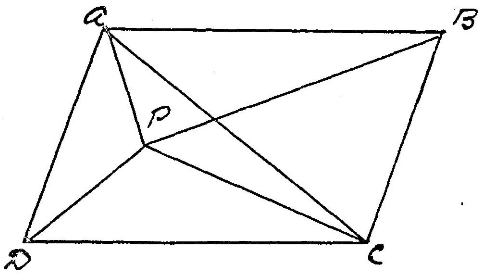
and (3)

$$GF \cdot FB + \overline{FB}^2 = EF \cdot GF + GF \cdot FB$$

$$\overline{FB}^2 = EF \cdot BF$$

Q.E.D.

Theorem XV. If ABCD is a parallelogram and P any point, the triangle PAC equals the triangle PAB minus the triangle PAD if P is within the angle BAD or that which is vertically opposite to it, and triangle PAD plus triangle PAB equals triangle PAC if P has any other position.



To Prove

$\triangle PAC = \triangle PAB - PAD$

Proof

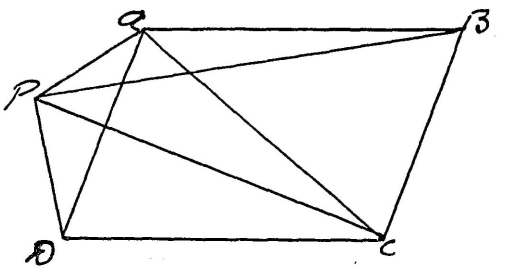
$\triangle PDC + \triangle APB = \frac{1}{2} ABCD$

$\triangle APD = \frac{1}{2} ABCD - \triangle APC - \triangle PDC$

$= \triangle PDC + \triangle APB - \triangle APC - \triangle PDC$

$\therefore \triangle APC = \triangle APB - \triangle APD$

Q.E.D.



To Prove

$\triangle PAC = \triangle PAD + \triangle PAB$

Proof:

$\triangle PAD = \triangle PAC + \triangle PDC - \triangle ADC$

$\triangle ADC = \frac{1}{2} ABCD = \triangle PAB + \triangle PCD$

$\therefore \triangle PAD = \triangle PAC + \triangle PDC - \triangle PAB - \triangle PCD$

$\therefore \triangle PAC = \triangle PAD + \triangle PAB$.

Q.E.D.

Theorem XVI. A parallelogram may be constructed in a given angle equivalent to a given rectilinear figure.

Theorem XVII. On a given line to construct a given rectilinear figure similar to and similarily placed to a given rectilinear figure.

SECTION II

The rectangle and the square lend themselves readily to problems of construction. More could be found than are included in this paper.

In the theorems on the rectangle there is a redundancy in the case of the proof of the converse pf PTOLEMY'S Theorem. This converse is included in Theorem VII but I have also given the proof as it is found in J. Hadamaid's Lecons de Géometice Elementaire, in Theorem IX.

Theorem I. A rectangle is the only parallelogram that may be inscribed in a circle.

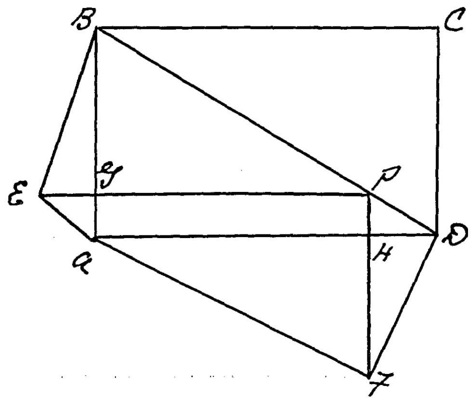
Theorem II. A rectangle may be constructed equivalent to the difference of two given squares.

Theorem III. A rectangle may be constructed equivalent to a given square when the sum of two adjacent sides equal a given quantity.

Theorem IV. A rectangle may be constructed equivalent to a given squate when the difference of the two adjacent sides equal a given quantity.

Theorem V. On AB and AD two adjacent sides of a rectangle, two similar triangles are constructed and perpendiculars are drawn from AB and AD to the angles they subtend intersecting at the point P. If AB and AD are homologous sides show that

P is in all cases on one of the diagonals of the rectangle.



To prove BPD is a straight line.

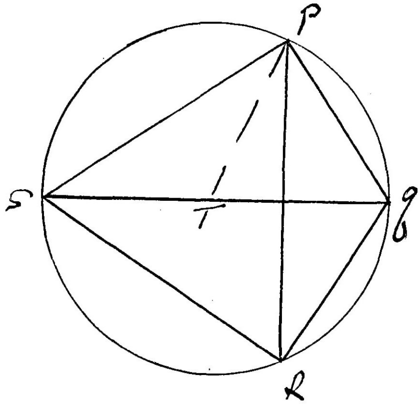
Proof: $\frac{EG}{GB} = \frac{FH}{HA}$ $\qquad$ $\frac{EG}{FH} = \frac{GB}{HA} = \frac{AJ}{HD}$

But AG PH and AH GP

$\therefore \frac{GB}{GP} = \frac{HP}{HD}$ $\quad \therefore$ BPD is a straight line and P is on BD.

Q.E.D.

Theorem VI. The sum of the rectangles contained by opposite sides sides of an inscribed quadrangle is equal to the rectangles contained by its diagonals.



To prove PS.RQ + PQ . SR = PR . SQ.

Proof:- Let PQ, RS be the quadrangle.

Construct $\angle$SPT = $\angle$RPQ and let PT cut SQ at T.

Now Δ s SPT, RPQ are equiangluar.

$\therefore$ PS : PR = ST : RQ

$\therefore$ PS . RQ = ST . PR

Again Δ s TPQ and SPR are equiangular.

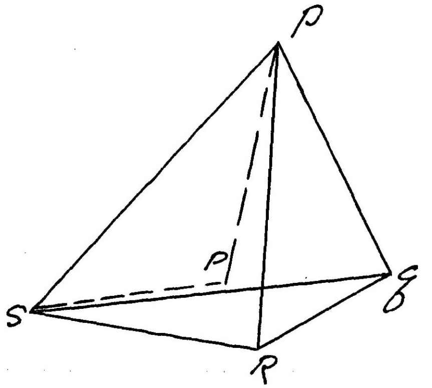
$\therefore$ PQ : PR = TQ : SR

PQ . SR = PR . TQ

$\therefore$ PS . RQ + PQ . SR = PR . ST + PR . TQ = PR . SQ.

Q.E.D.

Theorem VII. The rectangle contained by the diagonals of a quadrangle is less than the sum of the rectangles contained by its opposite sides unless the quadrangle is inscribed in which case it is equal to that sum.



Let PQRS be the quadrangle.

Make $\angle$ SPT = $\angle$ RPQ and $\angle$ PST = $\angle$ PRQ

Now s SPT, RPQ are equiangular.

$\therefore$ PS : PR = ST : RQ or PS . RQ = ST . PR

Also PT : PQ PS : PR or PT : PS = PQ : PR

and $\angle$ TPQ = $\angle$ SPR $\therefore$ $\triangle$s TPQ, SPR are equiangular.

$\therefore$ PQ : PR = TQ : SR

$\therefore$ PQ . SR = PR . TQ

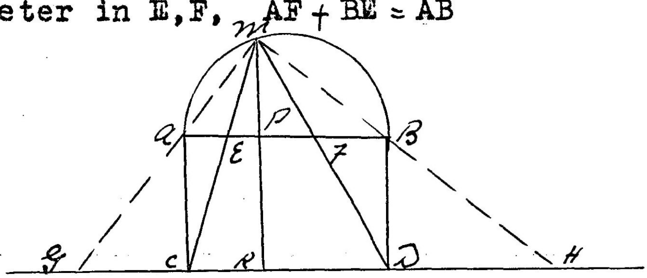
$\therefore$ PR . ST + PR . TQ = PS . RQ + PQ . SR

But SQ < ST + TQ unless STQ is a straight line.

$\therefore$ PR . SQ < PS . RQ + PQ . SR unless STQ is a straight line.

If STQ is a straight line $\angle$ QSP = $\angle$ QRP. $\therefore$ P. Q . RS are concyclic.

Theorem VIII. On the diameter AB of a semi-circle, construct a rectangle whose altitude AC is equal to the side of an inscribed square. If one joins the vertices C,D to any point M of the semicircumference by the lines CM, DM cutting the diameter in E,F, $\overline{AF}^2 + \overline{BE}^2 = \overline{AB}^2$



Proof: Draw the chords MA, MB and prolong these lines until they meet in G,H, CD prolonged. The angle GMH is inscribed in a semicircle.

$\therefore \angle$ SG and H are complementary and in consequence, the right s GCA, BDH are similar.

And $\frac{GC}{BD} = \frac{AC}{DH}$ or $2GC \cdot DH = 2\overline{AC}^2 \cdot \overline{CD}$

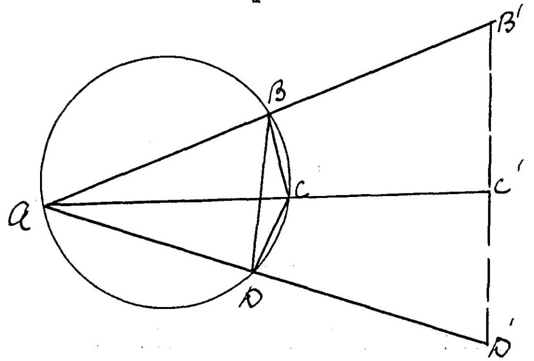
The segments GC, CD, DH being proportional to the segments AE, EF, FB. The preceding equality gives

$2\ AE \cdot BF = \overline{EF}^2$.

From $AF + BE = AB + EF$ one obtains $\overline{AF}^2 + \overline{BE}^2 + 2AF \cdot BE = \overline{AB}^2 + \overline{EF}^2 + 2AB \cdot EF$ or $\overline{AF}^2 + \overline{BE}^2 + 2AF \cdot BE = \overline{AB}^2 + 2(AE \cdot BF + AB \cdot EF)$. The identity $(a+b)(b+c) = ab + c(a+b+c)$ proves that $AF \cdot BE = AE \cdot BF + AB \cdot EF$

$\therefore \overline{AF}^2 + \overline{BE}^2 = \overline{AB}^2$

Theorem IX. The product of the sides AC and BD of an inscribed quadrangle ABCD is equal to the sum of the products of the opposite sides.



Let ABCD be inscribed in a circle.

Tp prove AC . BD = AD . BC + AB . CD.

Proof: Transform by inversion, using A as the center. The circle ABCD transforms into a straight line which passes through the points B^I; C^I; D^I the inverses of BCD. I C be the vertex of the quadrangle opposite A the points B and D will be on different sides of Ac and also the points B^I and D^I. The point C^I will lie between B^I and D^I and will have (in absolute value) $B^ID^I = B^IC^I + C^ID^I$ (I)

But if K is the power of the inversion $B^ID^I = DB \cdot \frac{K}{AB.AD}$

$B^IC^I = CB\ \frac{K}{AB.AC}$ and $C^ID^I = DC \cdot \frac{K}{AC \cdot AD}$

Substituting in these values in (I) and multiplying through by AB . AC . AD and dividing through by k, it follows AC.BD = AD.BC + AB.CD.

Q.E.D.

SECTION III

Theorem I. A square on a line is eqivalent to the squares on two parts together with twice the rectangle contained by the two parts.

Theorem II. In a given circle to inscribe a given square.

Theorem III. To describe a circle about a given square.

Theorem IV. To describe a square about a given circle.

Theorem V. A square is the only rectangle that can be described about a circle.

Theorem VI. The greatest rectangle that may be inscribed in a circle is a square.

Theorem VII. The square on the hypotenuse of a right triangle is equivalent to the sum of the squares on the other two sides

SECTION IV

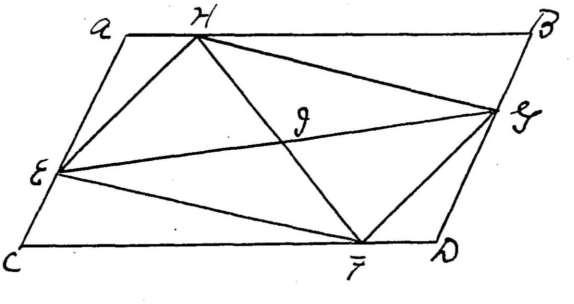
Theorem I. The diagonals of a rhombus are perpendicular to each other.

Theorem II. The diagonal of a rhombus bisects the angles through which it passes.

Theorem III. A rhombus is the only parallelogram that may be described about a circle.

Theorem IV. A rhombus may be constructed equivalent to any parallelogram.

Theorem V. Inscribe a rhombus in a parallelogram, so that one of the angular points of the rhombus may be at a given point in a side of the parallelogram.



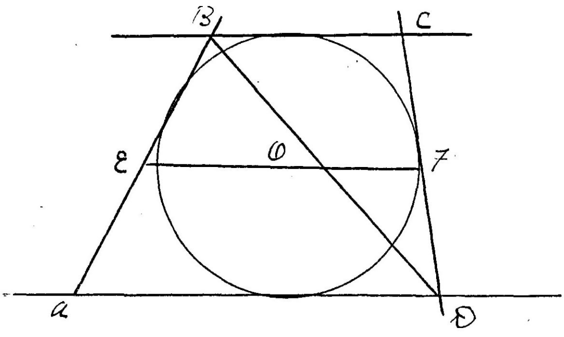
Let E be the given point.

Measure BG CE . Draw GE. Erect bisector H.F.

Draw HE,EF,FG and HG. The triangles may be asily proven equal.

SECTION V

Theorem I. If a trapeziod be described about a circle, a straight line drawn through the middle of the circle parallel to the bases and terminated by the two sides is equal to one-fourth the perimeter of the trapeziod.



To prove EF = I/4 (AB+BC+CD+DA)

Proof: EF = I/2 (BC+AD)

But BC + AD = AB + CD . ∴ EF = I/4 perimeter.
Q.E.D.

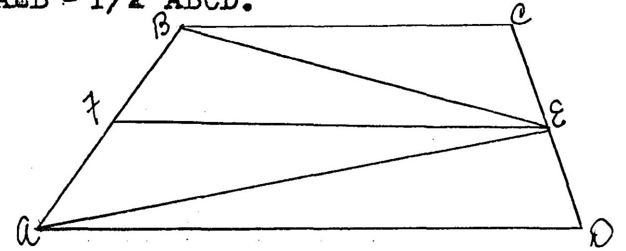
Theorem II. If a line parallel to the bases of a trapeziod it will divide the sides proportionally.

Use fig. in Theorem I.

To prove BE : AE :: CF : FD.

$\frac{BE}{AE} = \frac{BO}{OD}$ $\frac{BO}{OD} = \frac{CF}{FD}$ $\therefore \frac{BE}{AE} = \frac{CF}{FD}$

Theorem III. If ABCD is a trapeziod with BC parallel to Ad, E the mdd point of DC. Prove that △ AEB = I/2 ABCD.



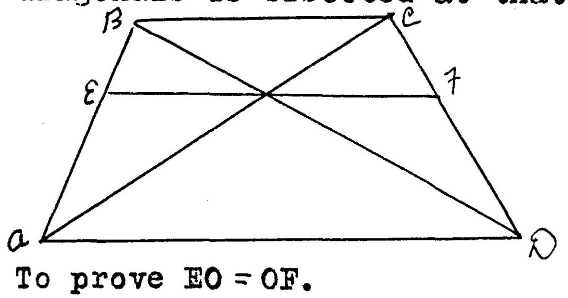
A. of ABCD = h(altitude) . FE

A. of △ BEF + △ AFE = I/2 h. FE.

∴ △ BEF + △ AFE = △ ABE = I/2 ABCD.

Q.E.D.

Theorem IV. A line drawn parallel to the bases of a trapeziod through the intersection of the diagonals is bisected at that point.



To prove EO = OF.

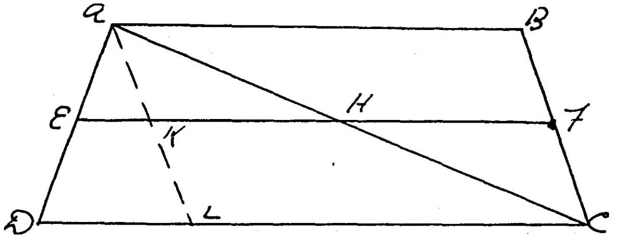
Proof: $\frac{EO}{AD} = \frac{BE}{BA} = \frac{CF}{CD} = \frac{OF}{AD}$

∴ $\frac{EO}{AD} = \frac{OE}{AD}$ ∴ $\underline{EO} = \underline{OE}$

Q.E.D.

V. In a trapezoid the angles adjacent to either of the non-parallel sides are supplementary.

The median of a trapezoid is parallel to the bases and equal to one half their sum



Given our trapezoid ABCD with EF the median.

To prove EF = I/2(AB+DC) and parallel to AB and CD. Draw AC, and AL parallel to BC. AL = BC,

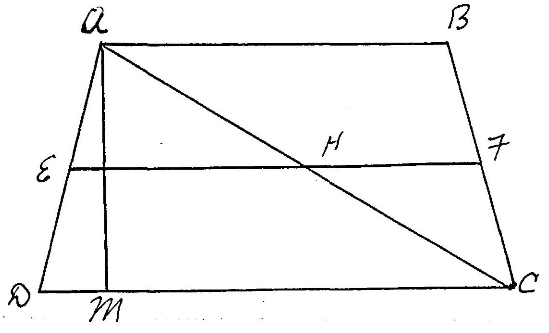
$\frac{AK}{KL} = \frac{FC}{BF}$, ∴ AK = KL and EF bisects AD and AL of ADL and is therefore parallel to DL.

But if EF is parallel to DC, it bisects AC of the ∢ ADC, therefore EH = I/2DC. In like manner FH = I/2AB

∴ EH + HF = I/2 (AB+CD)

Q.E.D.

Theorem VI. The area of a trapeziod equals the product of the median and the distance between the parallel sides.



Given the trapeziod ABCD with EF the median and AM the distance between the parallel sides AB and CD.

To prove: Area of ABCD = EF . AM.

Proof: EF = I/2(AB+DC)

Area of ABCD = area of ($\triangle$ ADC + $\triangle$ ABC)

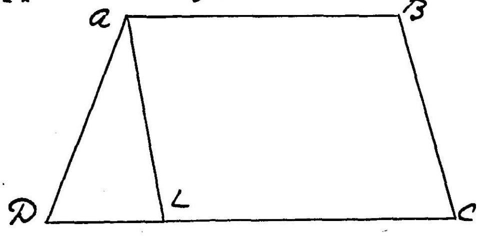
Area of $\triangle$ ADC = I/2 AM . DC

Area of $\triangle$ ABC = I/2 AM . AB

Area of ABCD = area of ($\triangle$ ADC + $\triangle$ ABC) = I/2 AM(AB+DC) AM.EF

Q.E.D.

Theorem VII. In an isosceles trapeziod the angles at each base are equal and the opposite angles are supplementary.



Let ABCD be an isosceles trapeziod.

To prove $\angle A = \angle B$, $\angle C = \angle D$

and $\angle A + \angle C = \angle B + \angle D = 2$ rt.$\angle$s.

Proof: Draw through A a line parallel to BC meeting CD in L.

$\triangle$ALD is an isosceles triangle, and $\angle ALD = \angle ADC$.

But $\angle ADC + \angle DAB = 2$ rt.$\angle$, since AB and CD are parallel and $\angle ALD = \angle BCD$. Substituting equals $\angle BCD + \angle DAB = 2$rt $\angle$.

In same manner $\angle ABC + \angle ADC = 2$ rt.$\angle$s.

But since $\angle BCD = \angle ALD = \angle ADC$ and $\angle BCD + \angle DAB = \angle ABC + \angle ADC$

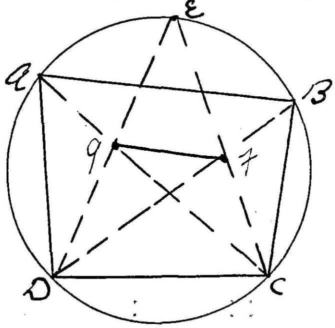
Subtracting $\angle BCD = \angle ADC$

$\angle DAB = \angle ABC$

Q.E.D.

SECTION VI

Theorem I. If ABCD is a quadrangle inscribed in a circle the straight lines CE, DE which bisect the angles ACB and ADB cut BD and AC at F and G respectively. Show that EF : EG = ED : EC



To prove Ef : EG :: ED : CE

Proof: In $\triangle$ s DFE and CGE, $\angle$ EDB = $\angle$ ACE and $\angle$ GEF is common. $\therefore$ EF : EG = ED : CE

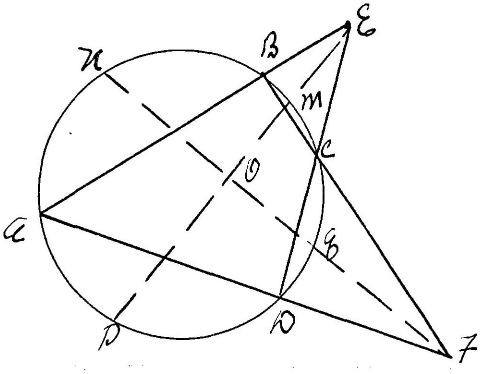
Q.E.D.

Theorem II. In a quadrangle ABCD if AD is the longest side and BC is the shortest, $\angle$ ABC > $\angle$ ADC and $\angle$ BCD > $\angle$ BAD.

Revolve ADC on AC as an axis. Then $\triangle$ ADC will include ABC as AD > AB and DC > BC. $\therefore$ $\angle$ D > $\angle$B.
In same way $\angle$C may be proved > $\angle$A.

Q.E.D.

Theorem III. The bisectors EP, FN of the angles formed by the intersection of the opposite sides of an inscribed quadrangle ABCD are perpendicular.



To prove: EP perpendicular to FN, where EP bisects $\angle$AED and FN bisects $\angle$AFB.

Proof: Arc PA - arc BM = arc DP - arc MC.

Arc AN - arc DQ = arc NB - arc CQ.

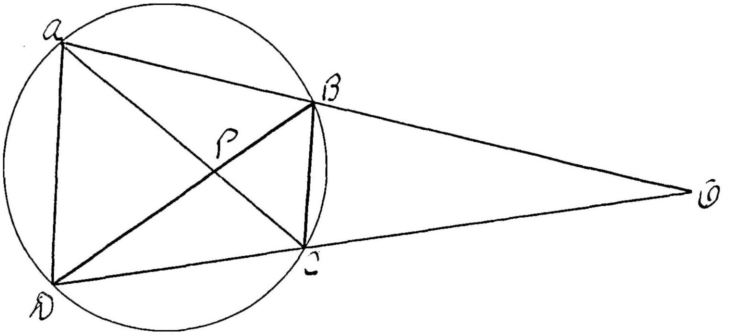
$\therefore$ arc Pa + arc AN - arc BM - arc DQ = arc DP + arc NB - arc MC - arc CQ or arc PN - arc BM - arc DQ = arc DP arc NB - arc MQ.

$\therefore$ arc PN + arc MQ = arc NM + arc PQ and $\angle$NOP = $\angle$NOM

But since $\angle$ NOP + $\angle$NOM were measured by half the circumference, $\angle$ NOP = rt$\angle$ and EP is perpendicular to FN.

Q.E.D.

Theorem IV. If ABCD is a quadrangle inscribed in a circle and the sides AB, CD when produced meet at O. Show that s AOC and BOD are equiangular.



To prove △ AOC and △ BOD equiangular.

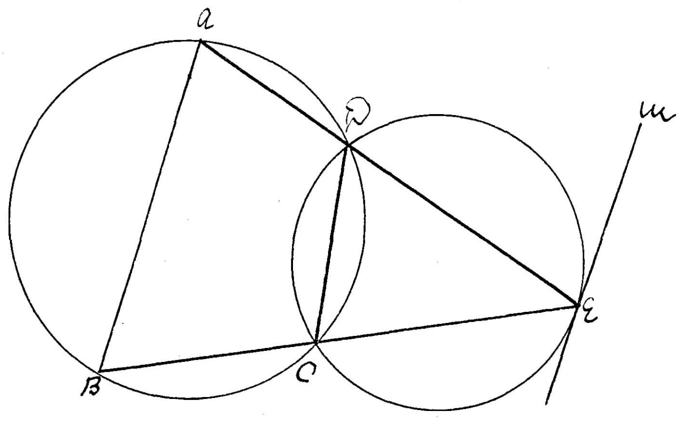
Proof ∠ O is common, ∠ OAC = ∠ BDC.

∴ △ s AOC and BOD are equiangular.

Q.E.D.

Theorem V. A quadrangle which can have one circle inscribed in it and another circumscribed about it, show that the straight lines joining the opposite points of contact of the inscribed circle are perpendicular to each other. Let ABCD be the quadrangle.

Theorem VI. A quadrangle ABCD is inscribed in a circle and AD, BC are produced to meet E. Show that the circle described about the triangle ECD will have the tangent at E parallel to AB.



To prove ME parallel to AB.

Proof: $\angle$ A = $\angle$ DCE.

But $\angle$ DCE = $\angle$ MED

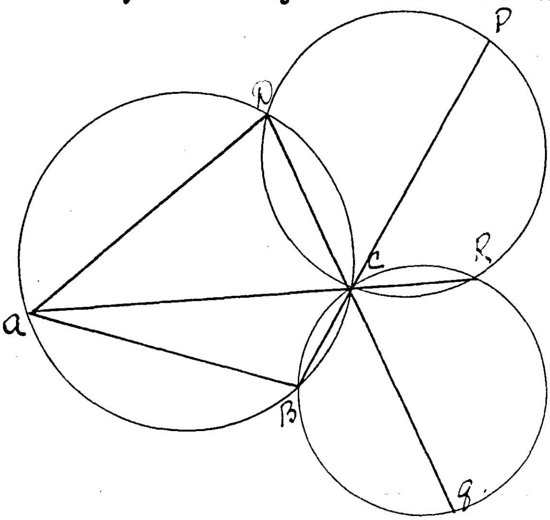
$\therefore$ $\angle$ MED = $\angle$ A and ME is parallel to AB.

Q.E.D.

Theorem . The sum of the opposite sides of a quadrangle circumscribed about a circle is equal to the sum of the other pair.

Theorem . The four sides of any quadrangle are togehter greater than the sum of the two diagonals.

Theorem VII. The opposite sides of an inscribed quadrangle meet at P, and Q and about the triangles so formed without the quadrangle circles are described meeting again at R. Show that P, R and Q are on a straight line.



To prove. PRQ is a straight line.

Proof: $\angle$ ADC + $\angle$ ABC = 2 rt. $\angle$s.

$\angle$ ADC + $\angle$ PBQ = 2 rt. $\angle$s.

$\angle$ ABC + $\angle$ QDP = 2 rt. $\angle$s.

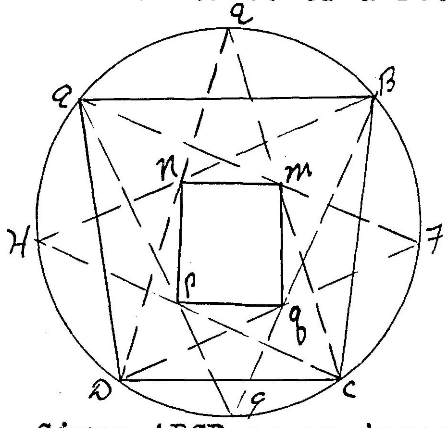
$\angle$ PDQ + $\angle$ CRQ = 2 rt. $\angle$s. $\angle$ QDP + $\angle$ CRP = 2 rt. $\angle$s.

$\therefore$ $\angle$ PBQ + $\angle$ CRQ + $\angle$ QDP + $\angle$ CRP = 4 rt. $\angle$s. = $\angle$ ABC + $\angle$ ADC + $\angle$ PBQ + $\angle$QDP. $\therefore$ $\angle$ CRQ + $\angle$ CRP = $\angle$ ABC + $\angle$ ADC = 2 rt. $\angle$s.

$\therefore$ PRQ is a straight line.

Q.E.D.

Theorem VIII. If ABCD is an inscribed quadrangle the centers M,N,P, and Q of the circles inscribed in the triangles ABC, ABD, ACD, BCD are the vertices of a rectangle .



Given ABCD as an inscribed quadrangle with E, F, G, and H the points where the arcs AB, BC, CD and DA are bisected, and N, M, Q and P the centers of circles inscribed in Δs ADB, ABC, BCD and CDA respectively.

To prove: MNPQ a rectangle.

Proof: $\angle$ EIF is measured by I/2(arc EB + arc BF + arc AD + arc DH) = I/4 (arc AEB + arc BFC + arc CQD + arc DHA) = I/4 circ.

$\therefore$ EG is perpendicular to HF.

Now E is the center and NE is the radius of a circle upon the line joining the incenter of ABD to the ex-center of the circle tangent to AB, as a diameter. This circle will pass through A, N, and B.

Considering the triangle ABC, the circdle with center at E and radius ME willpass through A, B, and M. Therefore NE = ME.

But EG bisects $\angle$ DEC.

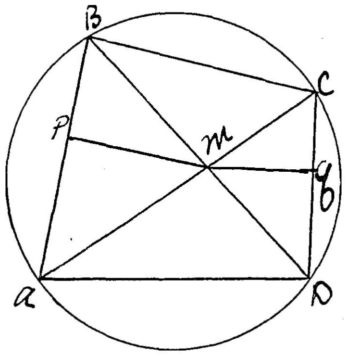
And EG must be perpendicular to MN.

In the same way FH can be proven perpendicular to MQ and NP.

But EG has been proven $\perp$ to FH.

$\therefore$ MNPQ must be a rectangle.

Theorem IX. In all inscribed quadrangles, ABCD the distance MP, MQ from the point of intersection of the sides AC and BD to the opposite sides AB, CD are proportional to these sides.



Given MP the ⊥ on CD from M and MP the on BA from M.

To prove: $\frac{MP}{MQ} = \frac{AB}{CD}$

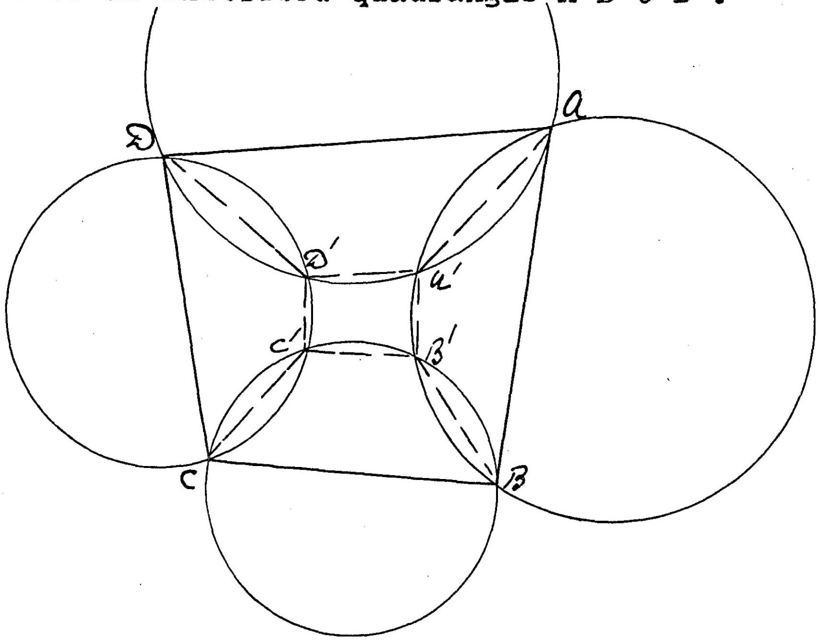
Proof: In the △ s AMB and CMD ∠ BAC = ∠ BDC, ∠ BMA = ∠ CMD.

Therefore △ AMB is similar to △ CMD and

$\frac{MP}{MQ} = \frac{AB}{CD}$

Q.E.D.

Theorem X. The circumferences which have for chords the sides of a quadrangle ABCD inscribed in a circle, give by their new intersections the locus of an inscribed quadrangle $A^IB^IC^ID^I$.



To prove: The sum of the angles about C are $\angle D^IC^IB^I + \angle B^IC^IC + \angle D^IC^IC = 4$ rt. $\angle$s.

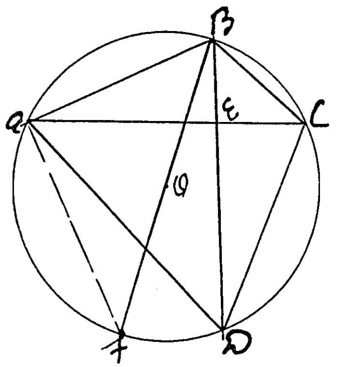
But $\angle B^IC^IC + \angle B^IBC = 2$ rt. $\angle$s

and $\angle D^IC^IC + \angle D^IDC = 2$ rt. $\angle$s.

Therefore $\angle D^IC^IB^I = \angle B^IBC + \angle D^IDC$. In the same way one can prove $\angle D^IA^IB^I = \angle B^IBA + \angle D^IDA$. Adding one obtains $\angle D^IC^IB^I + \angle D^IA^IB^I = \angle CBA + \angle CDA = 2$ rt. $\angle$s.

Q.E.D.

Theorem XI. In all quadrangles inscribed in a circle whose sides AC and BD are perpendicular the sum of the squares on the opposite sides is equivalent to the square on the diameter.



Given AC $\perp$ to BD and ABCD inscribed in circle O.

To prove: $\overline{AB}^2 + \overline{CD}^2 = \overline{BF}^2$

Proof: Since $\angle$ BEC = rt. $\angle$, arc BC + arc AD = semicircumference.

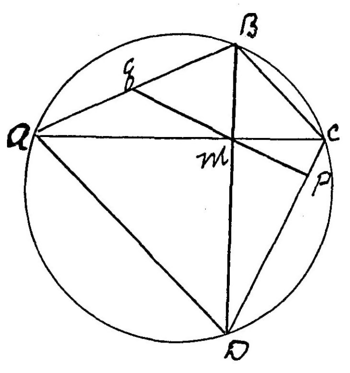
Draw the diameter BF. Since $\angle$ BAF = rt. $\angle$ arc AF = arc DC and AF = DC.

But $\overline{AB}^2 + \overline{AF}^2 = \overline{FB}^2$

$\therefore \overline{AB}^2 + \overline{DC}^2 = \overline{FB}^2$ Q.E.D.

Corollary- In all quadrangles inscribed in a circle, and whose sides AC and BD intersect at rt. $\angle$s, the sum of the squares of the sides eqal to eight times the square of the radius.

Theorem XII. In all inscribed quadrangles having the two sides AC and BD perpendicular, the perpendicular MP to one of the sides CD through the point of intersection of AC and BD, passesthrough the middle point of the opposite side.



Given AC $\perp$ to BD and PQ $\perp$ to CD through M.

To prove QP bisects AB.

Proof: $\angle$ ACD is the complement of $\angle$ CMP and therefore of its equal $\angle$AMQ.

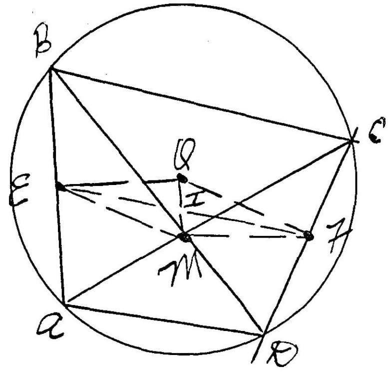
Therefore $\angle$ACD= $\angle$BMQ. But $\angle$ACD = $\angle$ ABD.

$\therefore$ $\angle$ BMQ = $\angle$ABD and QM = QB.

But $\triangle$ MAB is a right $\triangle$ and QA = QM.

$\therefore$ QA = QB.

Theorem XIII. In an inscribed quadrangle ABCD whose sides AC and BD intersect at right angles, the distance OE to the center from the sides AB, equals the half of the opposite side.



Let M be the intersection of AC and BD And OE be drawn ⊥ to AB.

To prove OE = I/2 CD.

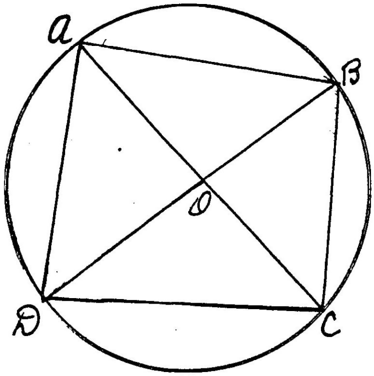
Proof: OE is parallel to the line which joins M to the middle point F. For the same reason OF is parallel to ME.

∴ MEOF is a parallelogram, and EO = MF. But CMD is a rt. △ and F is the midpoint.

∴ MF = I/2 CD.

Q.E.D.

Theorem XIV. The sides AC and BD, of the quadrangle ABCD inscribed in a circle, are to each other as the sum of the products of the sides which intersect at their extremeties.



Let ABCD be one quadrangle with AC and BD intersecting at O.

To prove $\frac{AC}{AB.AD + BC.CD} = \frac{BD}{AB.BC + CD.AD}$

Proof: OAD and BOC are similar.

$\frac{OA}{OB} = \frac{AD}{BC} = \frac{OD}{OC}$ and we can write

$$\frac{OA}{AB.AD} = \frac{OB}{AB.BC} \qquad \frac{OC}{BC.CD} = \frac{OD}{AD.CD}$$

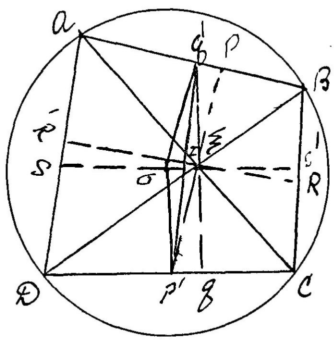
In the same way OAB and OCD are similar and

$\frac{OA}{OD} = \frac{AB}{CD}$ or $\frac{OA}{AB.AD} = \frac{OD}{AD.CD}$

$\therefore \frac{OA}{AB.AD} = \frac{OB}{AB.BC} = \frac{OC}{BC.CD} = \frac{OD}{CD.AD}$ adding the third and first and second and fourth

$\frac{AC}{AB.AD \quad BC.CD} = \frac{BD}{AB.BC \quad CD.AD}$ Q.E.D.

Theorem XV. If the quadrangle ABCD be turned about the fixed point M in such a manner that the sides AC, BD, intersecting at right angles, always cross at the point M: The mid point of AB, of BC, of CD, and of AD describe a circumference having its center at a point I of MO; Each median as EF is a diameter of this circumference; the radius R of this, is given by the formula $4R'^2 = 2R^2 - \overline{MO}^2$



To prove: $IQ' = IP' = IR' = IS' = 1/2\ \sqrt{2R^2 - \overline{Mo}^2}$

Proof: Let $AB = a$ $Cd = c$ Then $2R = \sqrt{a^2 + a^2}$ But $\overline{OQ}_1^2 + \overline{OQ}^2 = 2\overline{QI}^2 + 2\overline{MI}^2$

We have proven $OQ' = 1/2$ CD and $OQ = 1/2$ AB

Substituting in these values

$1/4(\overline{CD}^2 + \overline{AB}^2) = 1/2\ \overline{Q'Q}^2 + 1/2\ \overline{MO}^2 \quad R^2 = 1/2\ \overline{Q'Q}^2 + 1/2\ \overline{MO}^2$

$\therefore$ When $\overline{Q'Q}^2 = 4R'^2 = 2R^2 - \overline{MO}^2$

But $Q'Q$ is the diagonal of the parallelogram

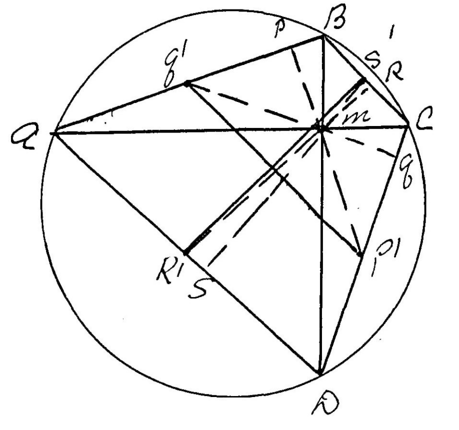
$Q'MQO$, and therefore is bisected by MO and $QI = Q'I$.

In the same way $R^I R^{I_2}$ can be proven equal to $2R^2 - \overline{MO}^2$ and therefore equal to $Q^I Q$. And $R^I S^I$ is bisected in the parallelogram $R^I MOS^I$ by MO at I and $RI = IS$.

$\therefore R^I S^I = Q^I Q$

and $RI = IS = Q^I I = QI = I/2 \sqrt{2R^2 - \overline{MO}^2}$

Theorem XVI. Let ABCD be a quadrangle in which the sides AC and BD intersect at right angles? If one projects on the sides the intersect M of AC and BD, and of one prolongs this line of projection until it meets the side opposite to the side perpendicular to this line: the eight points so obtained will lie on the same circumference; the center and radius of this circumference depend solely on the circumference ABCD and the point M.

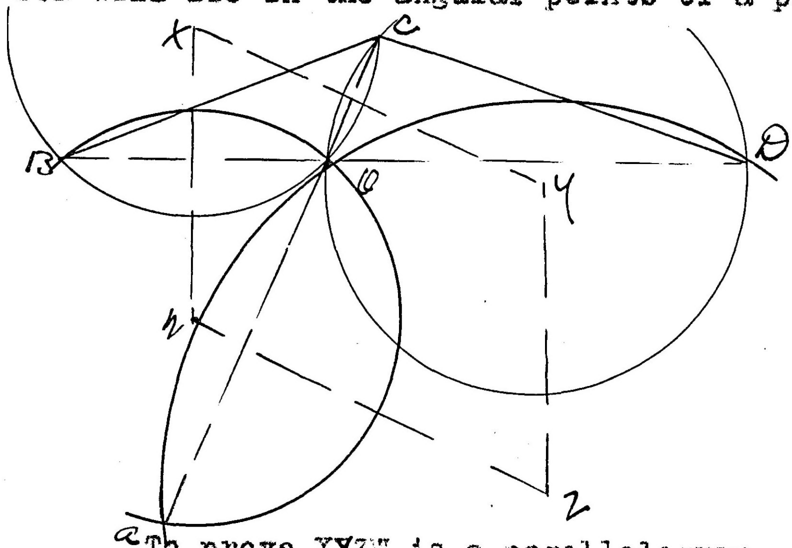


To prove R^I, S, P^I, Q, S'R, P, and Q^I lie on the same circumference. And that this circumference depends on circle ABCDand the point M.

Proof: Draw $PMP^I \perp$ to AB and $QMQ^I \perp$ to CD. By Brahmegupta's theorem the points P', Q' are

are the midpoints of CD and AB respectively.
The four points P, Q, P', Q' lie on the circumference described with the median P'Q' as a diameter. Another diameter is the median R'S', and the circumference PQP'Q' contains the four points R,S,R',S'.

Theorem XVII. The diagonals of a given quadrangle ABCD intersect at O show that the centers of circles described about the s AOB, AOD, OBC, OCD will lie in the angular points of a parallelogram.



To prove XYZW is a parallelogram.

Proof: $\triangle$ XCO is isosceles triangle.

$\triangle$YCO is isosceles triangle.

$\therefore$ XY is $\perp$ to CO. In the same way WZ is $\perp$ to CA.

$\therefore$ WZ and XY are parallel.

$\triangle$ XBO is isosceles triangle.

$\triangle$ BOW is isosceles triangle.

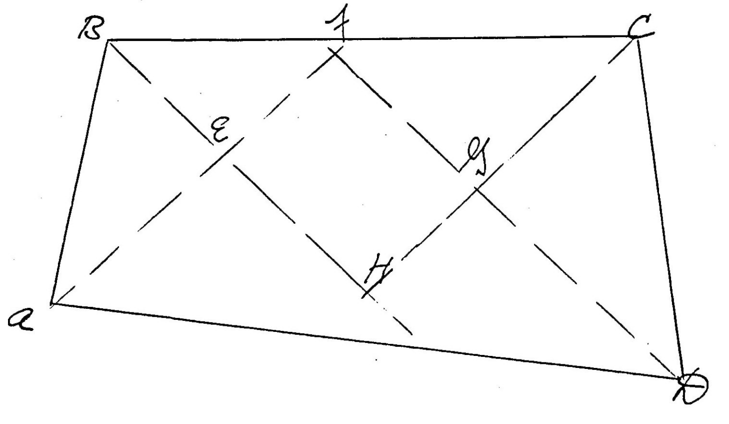
$\therefore$ XW is $\perp$ to BO. And in the same way WZ is $\perp$ to BD.

$\therefore$ XW and YZ are parallel and XYZW is a parallelogram.

Q.E.D.

Theorem XVIII. If a quadrangle be bisected by one diagonal, the second diagonal is bisected by the first.

Theorem . To prove that the four lines bisecting the angles of a quadrangle form a quadrangle which may be inscribed in a circle.



To prove EFGH may be ibscribed.

Proof: $\angle A + \angle B + \angle C + \angle D = 4$ rt. $\angle$s.

$1/2(\angle A + \angle B + \angle C + \angle D) = 2$ rt. $\angle$s.

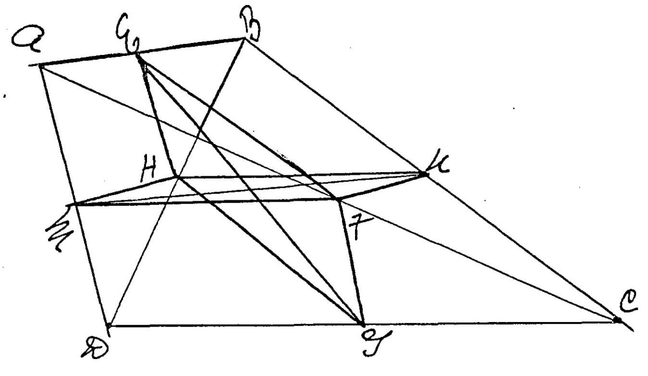
But in Δ a ABE and GCD

$1/2(\angle A + \angle B + \angle C + \angle D) = \angle BEA + \angle CGD = 4$ rt. $\angle$s.

(I) $\therefore$ $\angle BEA + \angle CGD = 2$ rt. $\angle$s.

In the same way it may be proved that $\angle BHC + \angle AFD = 2$ rt. $\angle$s. Substitute in (I) the equal angles, $\angle FEH + \angle FGH = 2$ rt. $\angle$s. And EFGH may be ibscribed.

Theorem XIX. Construct a quadrangle having given the length of the four sides AB, BC, CD and AD and one of the lines which join the middle points of the opposite sides.



Let ABCD be the desired quadrangle in which independent of the sides, EG is given which passes through the middle points of AB and CD.

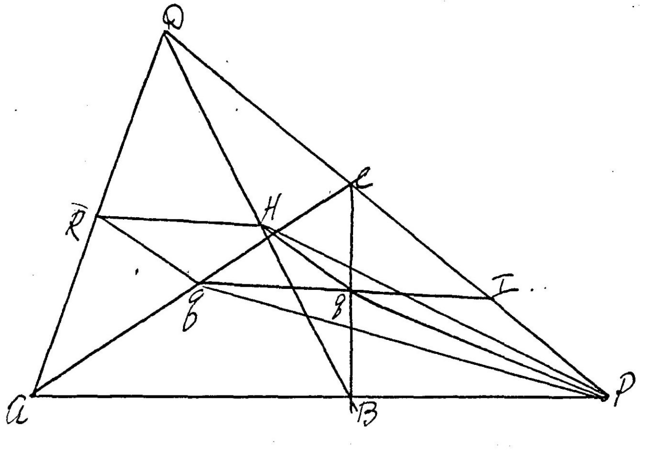
To construct ABCD. EFGH is a parallelogram, since EF and HG are parallel to BC and equal to half BC. Likewise EH and FG are parallel and equal to half of AD. Therefore given the length of AD and BC and the length EG, EFGH may be constructed.

In the same manner MHNG may be constructed.

But through the points M, E, N, and G, lines may be drawn parallel to FG, MH, HG and MF.

ABCD is therefore the desired quadrangle.

Theorem XX. If the opposite sides AB, CD of a quadrangle meet in P and if G, H be the middle points of the sides AC and BD the triangle PGH equals the quadrangle ABCD.



Let ABCD be one quadrangle, H the midpoint of BD, G of AC, and R and Q of AD and BC respectively.

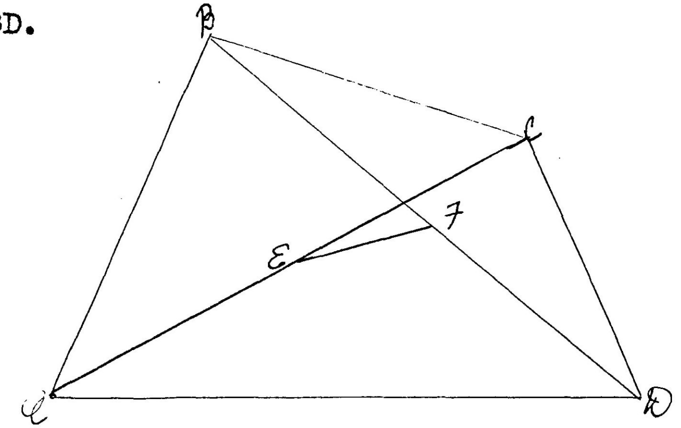
To prove $\triangle$ PQH = I/4 ABCD.

Proof: Draw QH, QQ, QP, RH, and RQ. Now since QG is parallel to AB if produced it will bisect PC. Joining the vertices of the $\triangle$s CGQ, PGQ on the same base GQ but on opposite sides, since CP is bisected by GQ, $\triangle$PGQ = $\triangle$CGQ = I/4 ABC. In the same way $\triangle$ PHQ = I/4 BCD. The parallelogram GQHR I/2(ABD - ABC)

$\therefore$ $\triangle$ QGH I/4 ABD - I/4 ABC: hence $\triangle$ PGH = I/4(ABC + BCD + ABD - ABC) = I/4 ABCD.

Q.E.D.

Theorem XXI. The sum of the squares of the sides AB, BC, CD, and AD, of a quadrangle is equal to the sum of the squares of the other two sides AC and BD plus four times the square of the line joining the middle points of AC, to BD.



Let ABCD be the quadrangle, E the middle point of AC and F of BD.

To prove $\overline{AB}^2+\overline{BC}^2+\overline{CD}^2+\overline{AD}^2=\overline{AC}^2+\overline{BD}^2+4\overline{EF}^2$

$2\overline{FB}.$

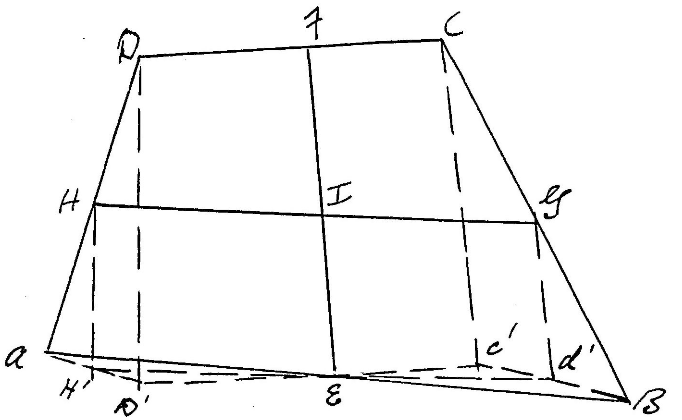
Proof: In $\triangle$ ABD $\overline{AB}^2+\overline{AD}^2=2\overline{AF}^2+2\overline{FB}^2.$ In $\triangle$ BCD

$\overline{BC}^2+\overline{CD}^2=2\overline{CF}^2+2\overline{FB}^2.$ $\therefore$ $\overline{AB}^2+\overline{BC}^2+\overline{CD}^2+\overline{DA}^2=2(\overline{AF}^2+\overline{CF}^2$

$4\overline{FB}^2=4\overline{AE}^2+4\overline{EF}^2+4\overline{FB}^2=\overline{AC}^2+\overline{BD}^2+4\overline{EF}^2.$

Q.E.D.

Theorem XXII. If a straight line EF divides protortionally two of the opposite sides of a quadrangle and if a second line GH divides the other two sides proportionally, each of these lines is divided by the other in the same ratio as the sides which determine them.



Given the quadrangle ABCD with the lines FE and HG so drawn that $\frac{AE}{BE} = \frac{DF}{CF}$ and $\frac{AH}{HD} = \frac{BG}{CG}$

To prove: $\frac{IE}{IF} = \frac{AH}{DH} = \frac{BG}{CG}$ and $\frac{IH}{IG} = \frac{AE}{BE} = \frac{DE}{CF}$

Proof: Draw $C^I D^I$ through E parallel to DC and DD^I, HH^I, $C^I C$, $G^I G$ parallel to EF.

By hypothesis $\frac{AE}{EB} = \frac{DF}{CF}$ But DF = D'E and CF = C'E and $\frac{AE}{EB} = \frac{D'E}{CF}$. In the ~~same~~ $\triangle$s AED', BEC'. $\angle$AED' $\angle$BEC' $\therefore$ AD', BC' are parallel. We have also by hypothesis $\frac{AH}{BG} = \frac{DH}{CG}$ or $\frac{AH'}{BG'} = \frac{D'H'}{C'G'}$ and since

A'D' and BC' are parallel, H' EG' are on a straight line and $\frac{EH'}{EG'} = \frac{AE'}{BE}$

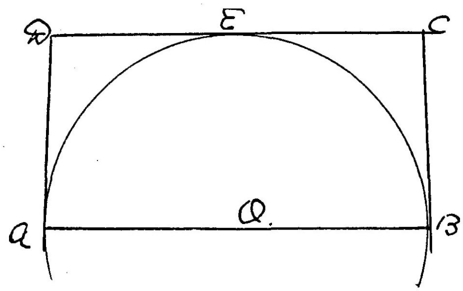
But HGH'G' is a parallelogram.

$$\frac{IH}{IG} = \frac{EH}{EG'} = \frac{AE}{BE} = \frac{DE}{CF}$$

In like manner

$$\frac{IE}{IF} = \frac{AH}{DH} = \frac{BG}{CG}.$$

Theorem XXIII. A quadrangle is bounded by the diameter of a circle, the tangents at its extremeties and a third tangent. Show that its area is equal to half that of the rectangle contained by the diameter and the side opposite to it.



To prove ABCD AB . $\frac{CD}{2}$

Proof: If area of ABCD $= \frac{AB \cdot CD}{2} = AB\frac{(AD + CB)}{2}$ AB.OE

But area of ABCD does AB . OE

$\therefore$ Area of ABCD $=$ AB . $\frac{CD}{2}$

Problem. Calculate the diagonals x,y of an inscribed quadrangle in terms of its sides, a,b, c, and d.

We have proved $xy = ac + bd$, andalso $\frac{x}{y} = \frac{ad + bc}{ab + cd}$

$\therefore x^2 = (ac + \frac{bd)(ad}{ab + cd} + bc)$. $y^2 = (ac + \frac{bd)(ab}{ad + bc} + cd)$

The theorems on the complete quadrangle will be calssified as follows:-

Section VII. Harmonic Properties.

Section VIII. Polar properties.

Section IX. Involutoric Properties.

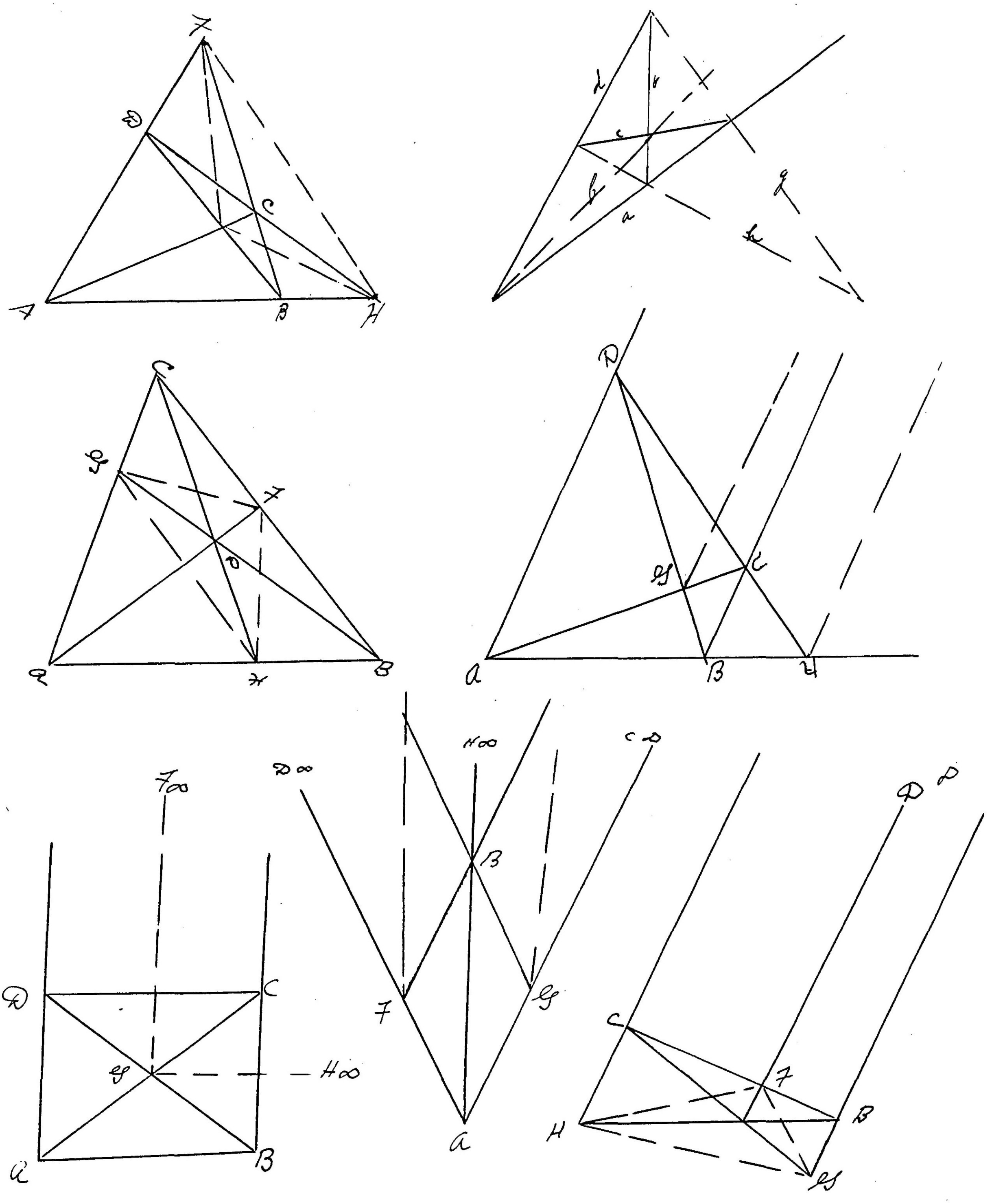
Section X. Quadrangle inscribed in a conic.

Section XI. Quadrangle circumscribed to a conic.

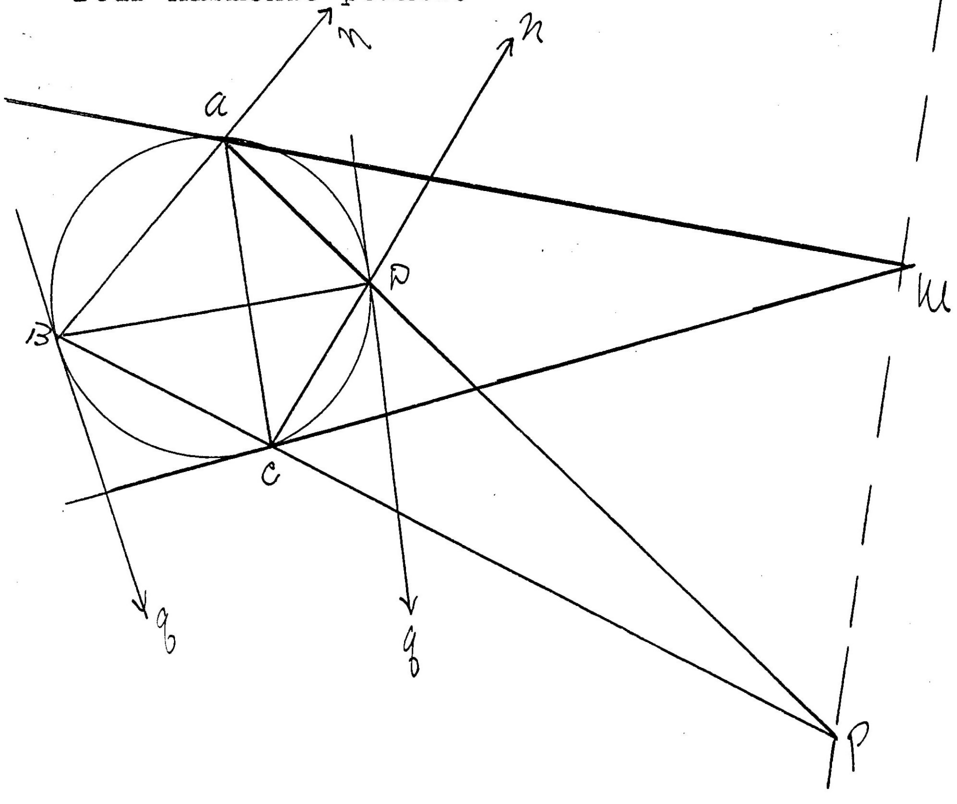
Section XII. Miscellaneous theorems.

By a quadrangle circumscribed to a conic will be meant one which has four of its six sides tangents to a conic.

Different forms of a complete quadrangle.



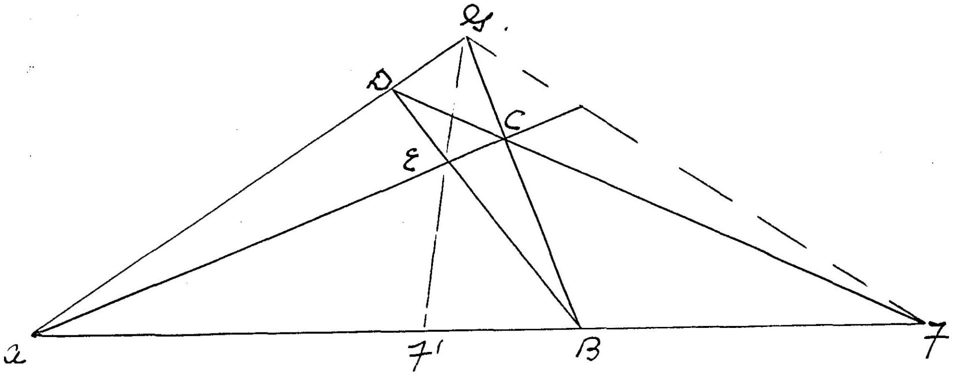
Theorem I. In every inscribed quadrangle the intersection of the opposite sides are the tangents at the opposite vertices are four harmonic points.



To prove M,N, P, and Q four harmonic points.

Proof: The three diagonal points of ABCD are NPR; NP is the polar of R and the tangents through A and C must intersect in M on NQ and also must tangents from B and D. But R(APNP) is a harmonic pencil and therefore N P M Q are four harmonic points. Q.E.D.

Theorem II. Any two vertices of a complete quadrangle are divided harmonically by the diagonal point lying on their junction line and the point in which their junction line is cut by the line joining the other two diagonal points.



Let ABCD be the quadrangle, EFG the diagonal triangle.

To prove: (FF', AB) is a harmonic range.

Proof: Since AC, BD and Ge are concurrent,

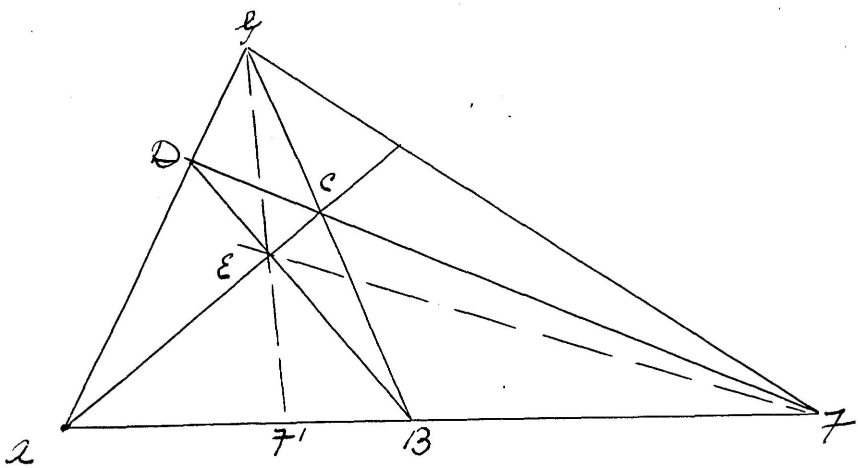
$$\frac{AF'}{F'B} \cdot \frac{BC}{CG} \cdot \frac{GD}{DA} = I$$

Also since FCD is a transversal of triangle GAB

$$\frac{AF}{BF} \cdot \frac{BC}{GC} \cdot \frac{GD}{AD} = I$$

$\therefore \frac{AF'}{F'B} = \frac{AF}{BF}$ $\quad\therefore$ (FF',AB) is a harmonic range

Theorem III. Any two sides of a complete quadrangle are divided harmonically by the diagonal line which passes through their intersection and the line joining their intersection to the remaining two diagonal point.



To prove that AD, BC are harmonic conjugates with respect to GE, and GF.

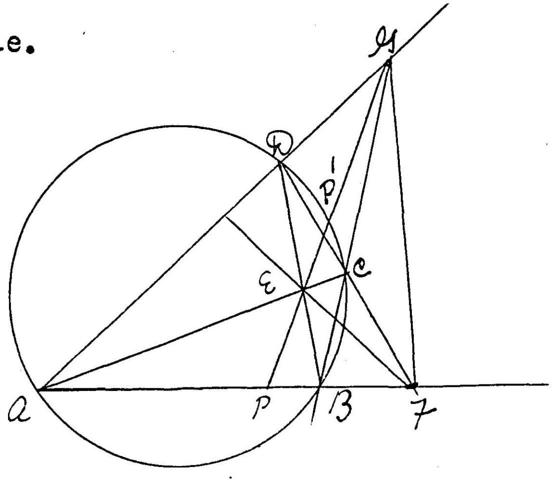
Proof: Since Ac, BD and GE are concurrent we have $\frac{AF'}{BF} \cdot \frac{BC}{GC} \cdot \frac{GD}{DA} = I$ and since FCD is a transversal of the triangle GAB, $\frac{AF}{BF} \cdot \frac{BC}{GC} \cdot \frac{GD}{AD} = I$

$\therefore \frac{AF'}{F'B} = \frac{AF}{BF}$

That is (FF'AB) is a harmonic range. $\therefore$ G(EF,AB) is a harmonic pencil and AD, BC are harmonic conjugates with respect to GE, and GF.

Q.E.D.

Theorem I. The centers of any quadrangle inscribed in a circle form a self conjugate triangle.

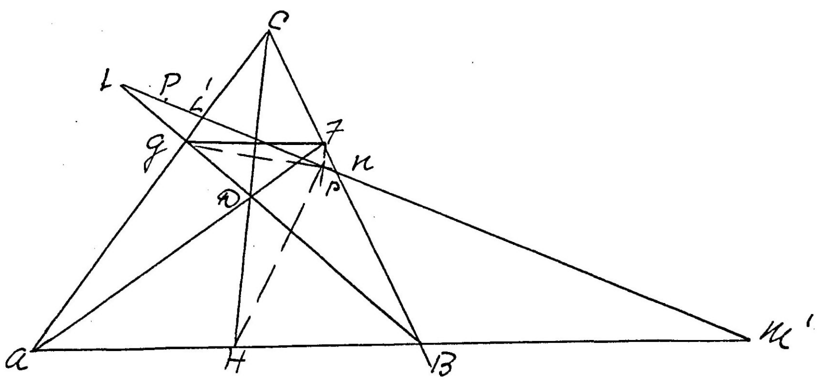


Let ABCD be any quadrangle inscribed in a circle and let E,F,G, be its centers. Then if AB, CD cut GE in P and P' it follows that(AB, PF) and (CD, P'F) are harmonic.

∴ GE is the polar of the point F. Similarly EF, FG are the polars of G and E respectively.

∴ EFG is a self conjugate triangle with respect to the circle.

Theorem II. In a plane the polar of a point P' with respect to the three pairs of opposite sides of a complete quadrangle passes through the same points.



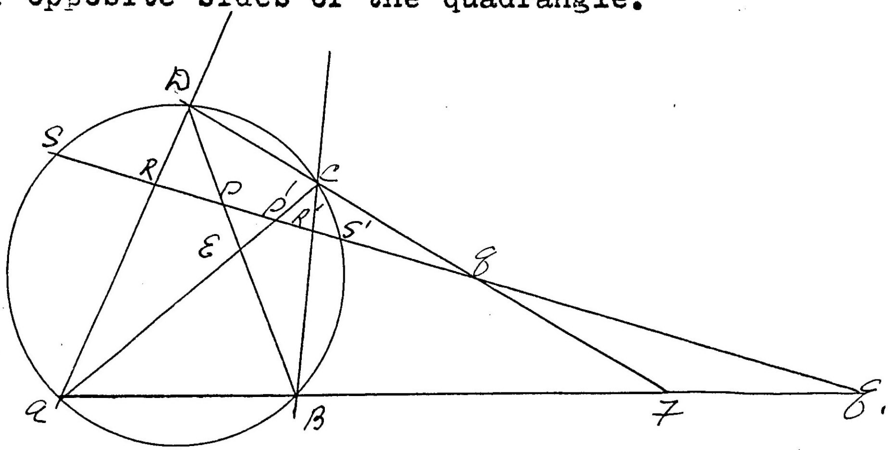
Let AB, CD intersect in H.

AC, DB intersect in Q and

AD, BC intersect in F.

Indicate by P' the point common to the polars, L, M of P respectively to the pairs of opposite sides of ABCD concurrent in F and G. The points P and P' divide harmonically the two pair of points L, L', M, M' determined by the line PP' with the pairs of opposite sides which pass through F and G and therefore are the double points of the involution LL', M M'. But NN' is also a pair of this involution of which also points NN' are divided harmonically by P, P' and at the same time the polar u of P with respect to the pair of opposite sides concurrent in H passes through P'.

Theorem I. If a quadrangle be inscribed in a circle any striaght line will be cut in involution by the circle and the three pairs of opposite sides of the quadrangle.



Let ABCD be a quadrangle inscribed in a conic and let any straight line cutting the sides AC, BD in P and P', sides CD, AB in Q and Q'; the sides AD, BC in R and R', and the circle in S and S'.

To prove: (PP', QQ', RR', SS') will be in involution.

Proof: Let AC and BD intersect at E. Then since the $\angle PAR = \angle R'BP'$ $\frac{RP}{AR} \sin RPA = \frac{P'R'}{BR'} \sin BP'R'$

$$\therefore \frac{AR}{RP} \cdot \frac{P'R'}{BR'} \cdot \frac{PE}{EP'} = 1$$ Similarily

$\angle RPP' = \angle PCR'$ and $\frac{RD}{RP'} \cdot \frac{PR'}{R'C} \cdot \frac{EP'}{EP} = I$

Hence $AR \cdot RD \cdot BR' \cdot R'C = RP \cdot RP' \cdot PR' \cdot P'R'$

But since ARD, SRS' are chords of the circle

$AR \cdot RD = SR \cdot RS'$ and similarly

$BR' \cdot R'C = SR' \cdot R'S'$.

$\therefore SR \cdot RS' : SR' \cdot R'S' = RP \cdot RP' : PR' \cdot P'R'$

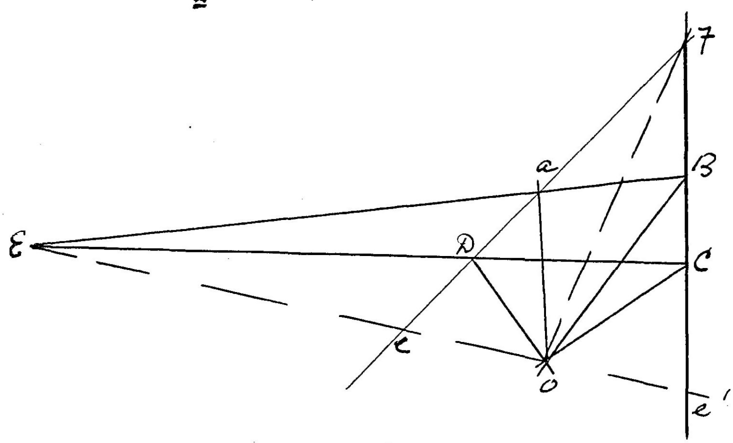
Hence (SS', PP', RR') are pairs in involution.

Similarly it may be proved,

(SS', QQ', RR') is in involution

Consequently (SS', PP', QQ', RR') is in involution.

Theorem II. The sixlines taken from a point to the four vertices of a quadrangle and to the points of intersection of the opposite sides form a pencil in involution.

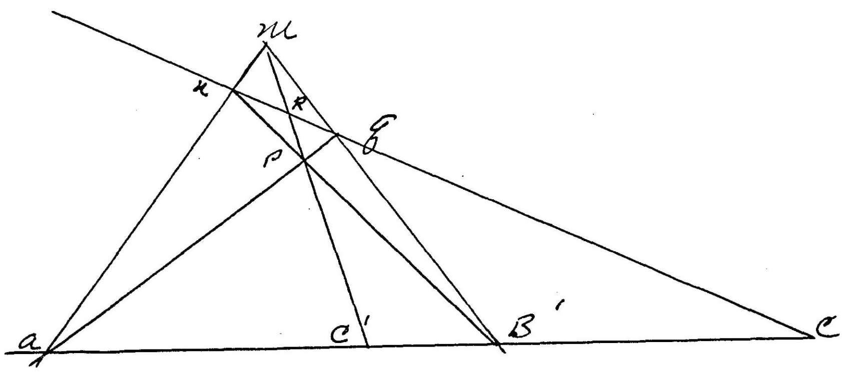


Let ABCD be the quadrangle. O be any point, let EO cut FD at E and FC at e'.

To prove O(AC, BD, EF) is a pencil in involution.

Proof: A,D,e, F and B,C,e', F are two ranges whose cross rates are equal since the three lines AB, DC, ee' meet in point E. It follows that the cross rates of the four lines OA, OD, OE, OF, equals the cross ratio of the lines OC,OB,OF,OE in such a way that the four mlines OA, OD, OE, OF correspond respectively to the four lines OC, OB, OF, OE. Then one has three systems of conjugate lines OA,OC', OD, OB', and OE, OF which are such that the four lines pertaining to the three systems have their anharmonic ratio equal and their four lines conjugate. Therefore the six lines form an involution.

Theorem III. Three pairs of opposite sides of a complete quadrangle ar cut by a transversal not passing through any vertex in three pairs of points belonging to one and the same involution.



To prove (AA', BB', CC') are in involution.

Proof: $MPRC' \overset{K}{\barwedge} ABCC'$

$MPRC' \overset{Q}{\barwedge} B'A'CC'$

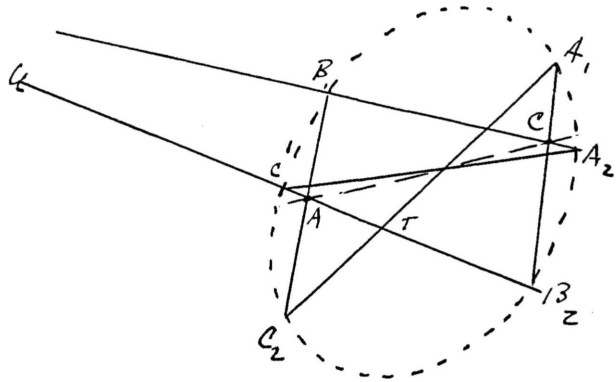
$\therefore$ $(ABCC') \barwedge (B'A'CC')$

But $(B'A'CC') = (A'B'CC')$

$\therefore$ $ABCC' \barwedge A'B'C'C$.

$\therefore$ AA', BB', CC', are in involution.

Theorem I. If the vertices of a complete quadrangle are on a conic which meets a line in two points, the latter are a pair in the involution determined on the line by the pairs of opposite sides of the quadrangle.



Let $A_I A_2 B_I C_2$ be the vertices of the quadrangle and let the line 1 meet the conic in B_2, C_I. This quadrangle determines on the line an involution in which S,H,T, and U are conjugate pairs. But the quadrangle $A_I A_2$ BC determines Q(B_zST,C,AU). Hence the two quadrangles determine the same involution on the line and therefore B_2, C_I are pair of the involution determined by the quadrangle $A_I A_2 B_I C_2$. Since the quadrangle $A_I A_2 B_I C_2$ and $A_I A_2$ BC determine the same involution on the line when the latter is a tangent to the conic, we have a special case of the above theorem:

Corollary: If the vertices of a complete quadrangle are on a conic the pairs of opposite sides meet the tangents at any other point in pairs of an involution of which the point of contact of the tangent is a double point.

Theorem II. If ABCD be any complete quadrangle whose six sides AB, AC, BC, AD, BD, CD are cut by an arbitrary straight line a in the points P,Q,R,ST,V; and let E,F,H,K,L,M, be the harmonic conjugates of these points with respect to the pairs of vertices of the quadrangle so that AEBP, AFCG etc. are harmonic ranges. Then a conic may be passed through the six points E,F,K,H,L,M on which shall lie the three points of intersection x,y, and z of the pairs of opposite sides of the quadrangle.

If a be the infinitely distant line of the plane the harmonic conjugates become the bisectors of the sides of the quadrangle and if at the same time the quadrangle be such that the three pair of sides intersect at right angles the conic becomes a circle.

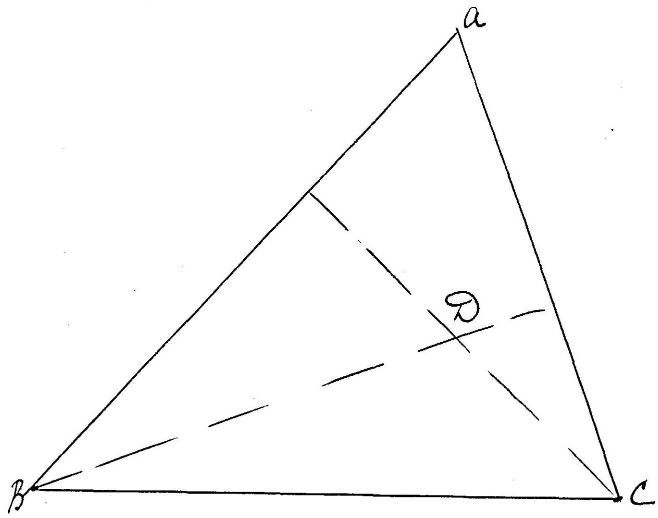
Since the line a may have any position in the plane it determines the doubly infinite system of conic through the three points x,y, and z.

Theorem III. If two quadrangles have the same harmonic points, then their eight vertices lie on a conic; as a particular case, if any three vertices of the vertices are collinear the eight vertices lie on two lines.

Let ABCD and A'B'C'D' be two given quadrangles and vuw be three common harmonic triangle.

If no three of the right vertices lie on a line we can draw a conic through any five say A',B',C',D' and A. Then from the inscribed quadrangle A'B'C'D' we see that U V W is a self conjugate triangle with respect to the conic. Also by hypothesis U V W is the harmonic triangle of the quadrangle ABCD. Hence B is such that W A U B is harmonic; hence B is on the conic, for A is on the conic and W is the pole of U V; similarly C and D are on the conic. ABCD A'B'C'D' lie on the conic.

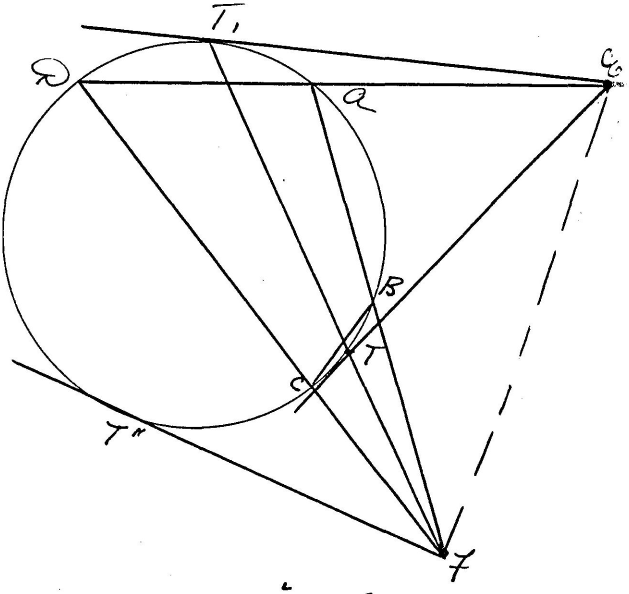
Theorem IV. Every conic circumscribing an orthogonal quadrangle is an equilateral hyperbola.



If ABCD be an orthogaonal quadrangle the line to infinity cuts the second pair of sides in an involution which is projected from a point in a circular involution. Therefore every conic circumscribed to this, determines upon the straight line to infinity two points in orthogonal direction to them, and for that reason is anequilateral hyperbola.

pole of line PR and therefore that line passing through V, the intersection of AC and BD. S is the pole of AD, Q is the pole of CB. Therefore O is the pole of SQ. Again as PR has been proved to be the polar of O, it must when produced pass through O', the intersection of AD and BC, but as O' is the pole of SQ, AO' is cut harmonically and therefore VA, VS, VD, VR form a harmonic pencil. Lastly as AC and BD are polars of the extremeties of the diagonal of PQRS this diagonal EF is the polar of their intersection V and therefore it must coincide with the diagonal O'O of the inscribed quadrangle ABCD.

Theorem V. If a quadrangle be inscribed in a circle, the square of the diagonal EF is equal to the sum of the squares of the tangents from its extremeties.



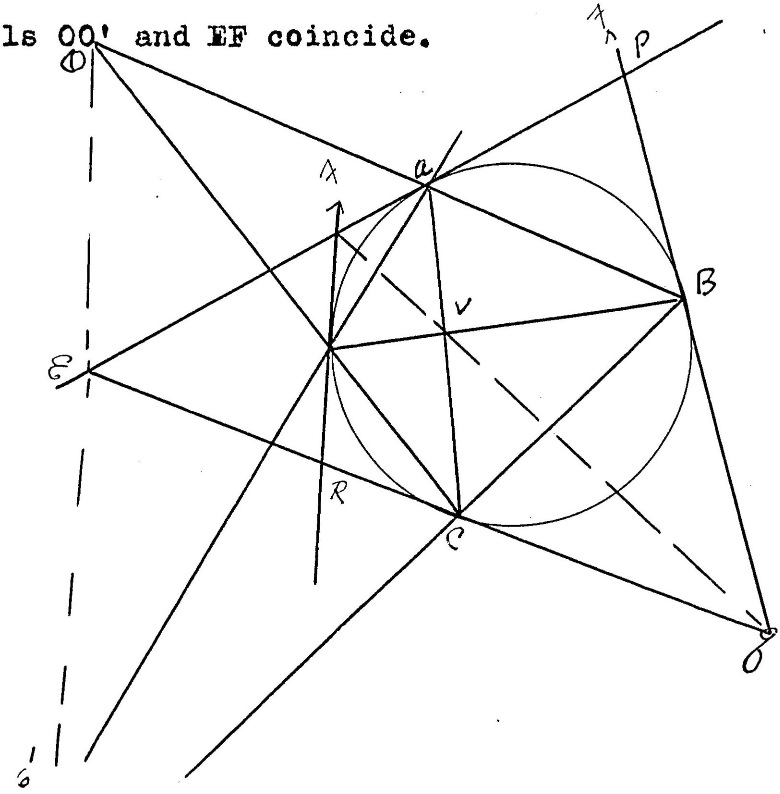
To Prove $\overline{EF}^2 = ET^2 + FT^2$

Proof: TT' is the polar of E and passes through F. $\overline{EF}^2 - \overline{ET}^2 = FT \cdot FT'$ the square of the tangent from F $= FT''^2$

$\therefore \overline{AF}^2 = \overline{ET}^2 + FT''^2$

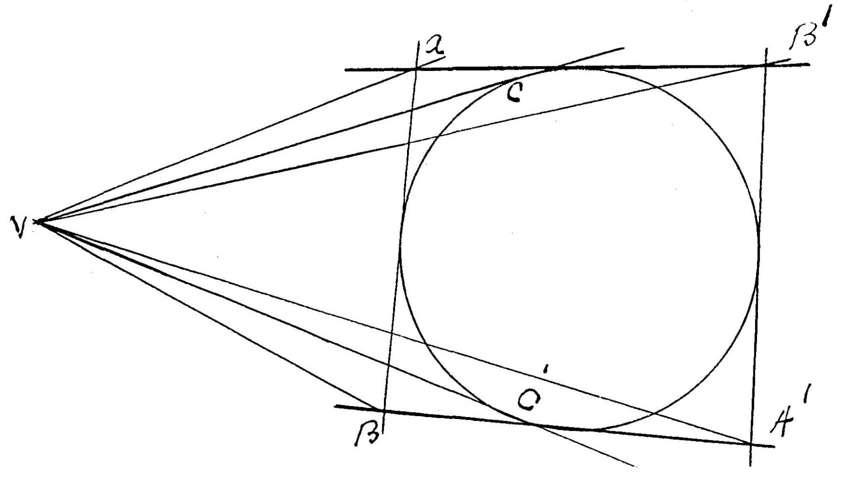
Q.E.D.

Theorem I. If a quadrangle be circumscribed to a circle and an inscribed quadrangle be formed by joining the successive points of contact, the sides AC, BD, PR, and SQ of the two quadrangles intersect on the same point and form a harmonic pencil and the diagonals OO' and EF coincide.



Let PQ RS be the circumscribed quadrangle. Let AB and CD meet at O. Then AB and CD are polars of P and R, the point O is the

Therem II. If a quadrangle be circumscribed to a circle two tangents from the same point to the vertices of the quadrangle form a pencil in involution.

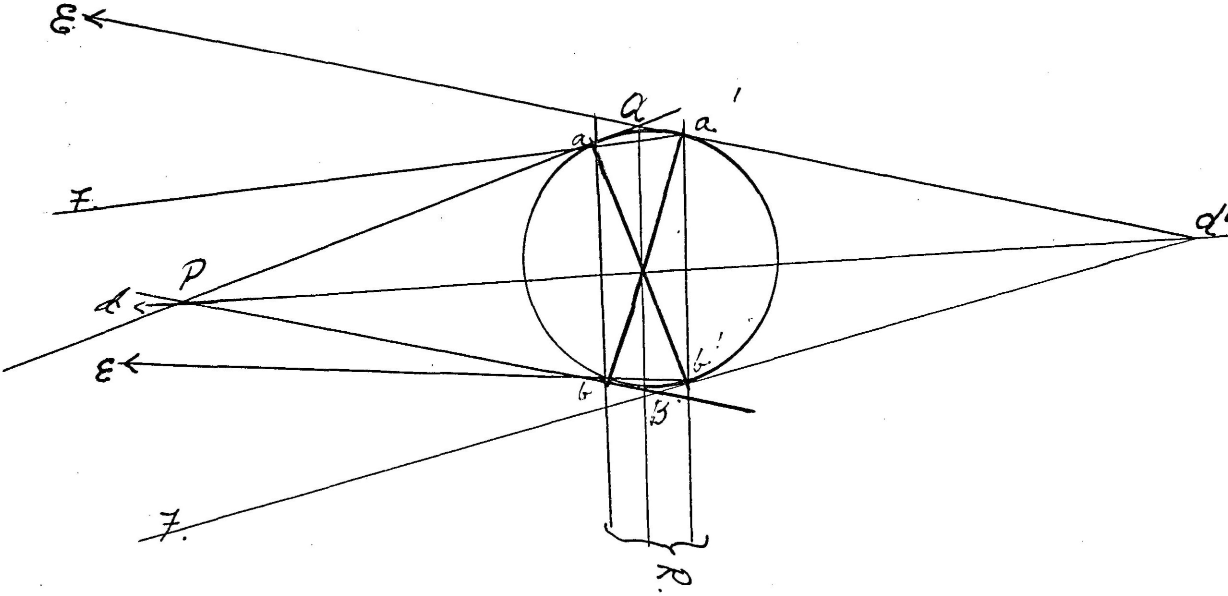


Let ABA'B' be the quadrangle, V any point and vc and vc' the tangents. Let the tangents bexproduced to meet A'B'. Then consider VC,VC', AB, and A'B' as four fixed tangents, the cross ratio of their intersection with the fifth tangent is a constant.

∴ V(ACC'B') = V(BCC'A') = V(A'C'CB) (AB' and A'B being taken successively as the fifth tangent)

Q.E.D.

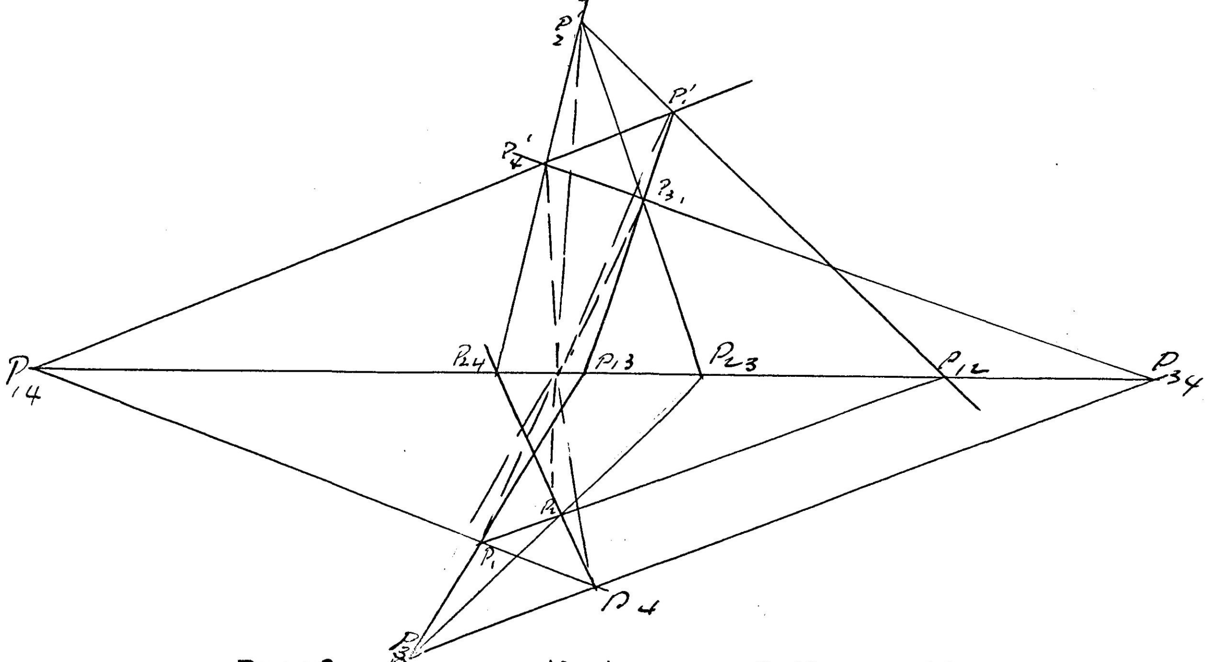
Theorem III. When a quadrangle is circumscribed to a circle one of the sides and the lines which join the points of contact of the opposite sides pass through the same point.



Let Ad Bd' be the circumscribed quadrangle, a'b, ab' be lines which join the points of contact of the opposite sides.

Lines A'b and ab' will intersect on the side AB since the two vertices A,B and the point of intersection of the two lines ab', a'b are three points which pertain to the polar of the point of intersection of the two lines aa', bb' For this reason the two lines ab' a'b will intersect on the second side dd'.

Theorem I. If two complete quadrangles are so situated that five sides of one meet five sides of the other in points of a line, the sixth side of the first meets the sixth side of the second in a point on the line.

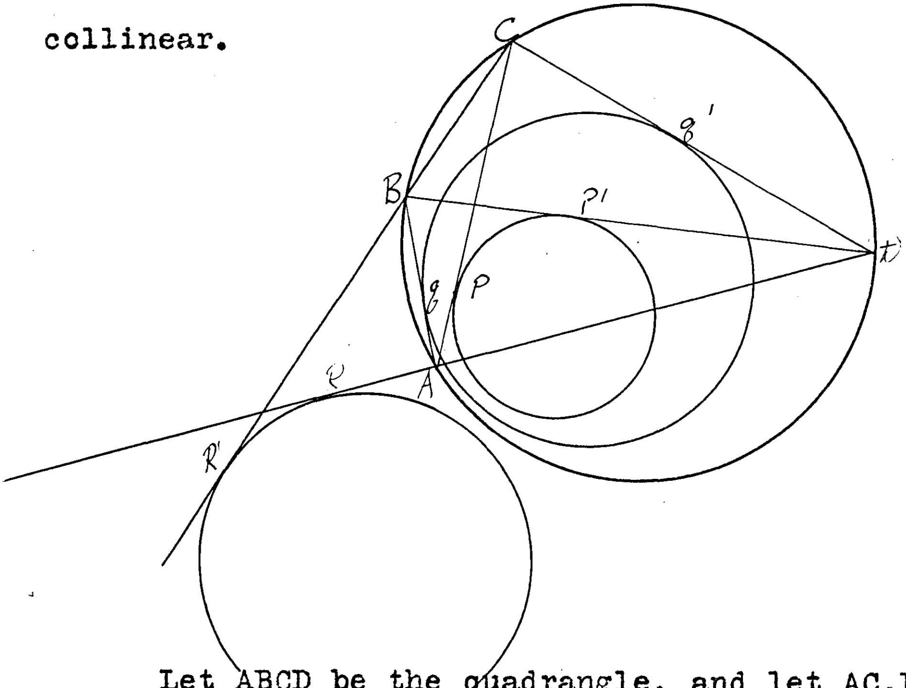


Proof: Suppose that none of the vertices or sides of one of the quadrangles coincide with any vertex or side of the other. Let P_IP_2 P_IP_3, P_IP4, P_2P_3, P_2P_4 be the five sides which by hypothesis meet their homologous sides $P_I'P_2'$ $P_I'P_3'$ $P_I'P_4'$ $P_2'P_3'$ P_2' P'_4 on points on 1. We must show that P_3P_4 and P_3' P_4' meet in a point on 1. The traingles $P_IP_2P_3$ and P_I' $P'_2P'_3$ are by hypothesis perspective from 1; as also the triangles $P_1P_2P_4$ and $P_1'P_2'P_3'$. Each pair is therefore perspective from a point and this point is in each case the intersection O of the lines P_1P_1' and P_2P_2' Hence the triangles $P_2 P_3P_4$ and $P_2' P_3' P_4'$ are perspective from O and their pairs of homologous sides intersect in the points of a line which is evidently l since it contains two points of l. But P_3P_4 and $P_3' P_4'$ are two homologous sides of thses last two triangles. Hence they intersect in a point of the line l.

If a vertex or side of one quadrangle coincides with a vertex or side of the other the proof is made by considering a third quadrangle whose vertices and sides are distinct from those of both of the others, and which has five of its sides passing through the five given points of intersection of homologous sides of the two given quadrangles. By the argument above its sixth side will meet the sixth side respectively of each of the two given quadrangles in the same point of l.

PONCELET'S THEOREM.

Theorem II. If a quadrangle be inscribed in a circle of a given coaxal system so that one pair of opposite sides touch another circle of the system then each pair of opposite sides will touch a circle of the system and the six points of contact will be collinear.



Let ABCD be the quadrangle, and let AC, BD touch another circle at points P_1 P'. Let PP' cut AB, CD in Q, and Q' and AD, Bc in R and R'.

Δ AQP, Δ DQ'P' are similar and

$\therefore$ AQ : AP = DQ' : DP' AP : AQ = Sin AQP' . SIn APQ and BP' : BQ = sin BQP' : Sin BP'Q. But the angles APQ, QP'B are equal and $\therefore$ AP : AQ = BP' : BQ. Hence AP : AQ = BP' : BQ = DP' : DQ'

CP : CQ'. Let x_1, x_2, x_3, x_4 denote the circles whose centers are A_1 B_1 C and D and whose radii are AQ, BQ, CQ', DQ' respectively. Now only one circle can be drawn coaxal with the given circles which will cut x_1 orthogonally. Let this circle be denoted by x (AX) : (BX) : (CX) : (DX) = AP : BP' : CP : DP'.

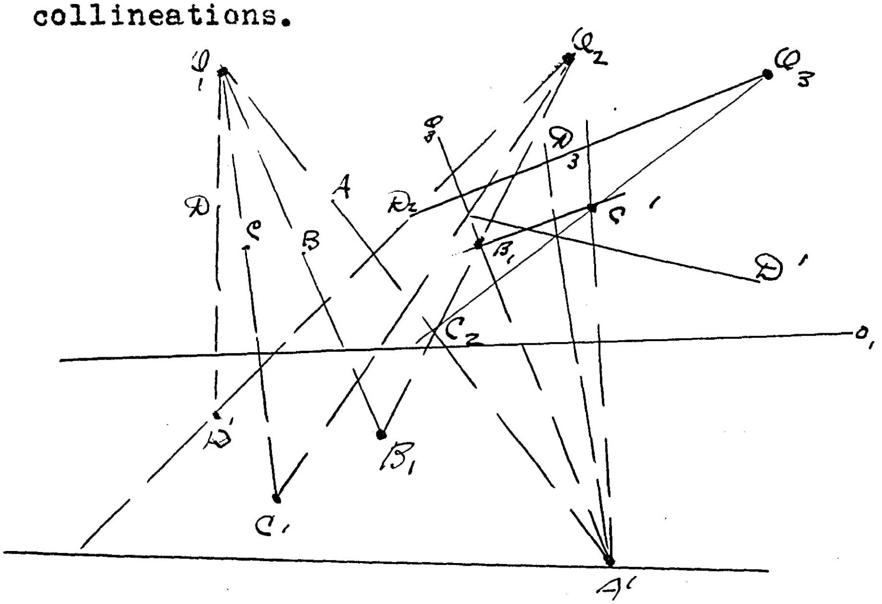
$\therefore$ (AX) : (BX) : (CX) : (DX) AQ : BQ : CQ' : DQ'

But (AX) = AQ $\therefore$ (BX) = BQ (CX) = CQ' (DX) = DQ'

Since (BX) = BQ it follows that x must cut x_x orthogonally. $\therefore$ x must pass through the limiting points of the circles x_1, x_2. But these circles touch at the point Q. Hence the circle x must touch AB at the point f. Similarly the circle x will cut orthogonally the circles x_3 x_4. Therefore since these circles touch at the point Q', the circles x must touch CD at Q'.

Thus the pair of connectors AB, CD touch the same circle of the coaxal systems at the points Q and Q'. In a similar manner it may be proved that the pair of connectors AD,BC will touch a circle coaxal with the given circles at the points R and R'

Theorem III. Any complete quadrangle of a plane can be transformed into any complete quadrangle of the same or different plane by a projective collineation which, if the quadrangles are in the same plane, is the resultant of a finite number of perspective collineations.



Proof: Let the quadrangles be in the same plane and let their vertices be A,B,C,D, and H' B' C' D'. we must show first that there exists a collineation leaving any three vertices say A' B' c' of the quadrangle A' B' c' D' in varient and transforming into the fourth,D', any other point D_3 not on a side of the triangle A'B'C'.

$A'B'C'$.

Let $\bar{D}$ be the intersection of AD_3, $B'D'$ and consider the homology with center A' and axis $B'C'$ transforming D_3 into $\bar{D}$. Next consider the homology with center B' and axis $C'A'$ transforming $\bar{D}$ into D'. Both these homologies exist by Theorem IX. The resultant of these two homologies is a collineation leaving fixed $A'B'C'$ and transforming D_3 into D_I.

Let O_I be any point on the line containing A and A' and let o_I be any line not passing through A or A'. There exists a perspective collineation , transforming A to A' and heaving O_I and o_I as centers and azis. Let $B_I C_I B$ be points such that $\pi_I(ABCD) = A'B_IC_ID_I$

In like manner let o_2 be any line through A' not containing B or B' and let O_2 be any point on the line B_IB'. Let π_2 be the perspective collineation with axis o_2, center O_2, and transforming B_I to B'. Let $C_2 = \pi_2(C_I)$ and $D_2 = \pi_2(C_I)$. Hence $\pi_2(A' B_IC_ID_I) = A' B' C_2D_2$

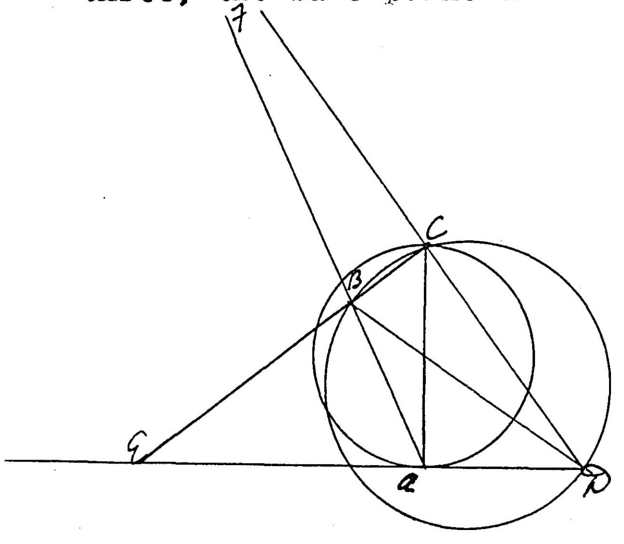
Now let O_3 be any point on the line C_2C' and let π_3 be the perspective collineation which has $A'B' = o_3$ for axis, O_3 for center, and transforms C_2 to C'. The existence of $_3$ follows as soon as we observe that C' is not on the line $A'B'$ by hypothesis, and C_2 is not on $A'B'$; because if sø, C_I would be on $A'B_I$ and therefore C would be on AB. Let $_3(D_2) = D_3$. It follows that $\pi_3(A'B'C_2 D_2) = A'B'C'D_3$

The point D_3 can not be on a side of the triangle $A'B'C'$ because then D_2 would be on o side of $A'B'C_2$, and hence D_I on a side of $A'B_IB_I$ and, finally D on a side of ABC. Hence, by the first paragraph of this proof, there exists a projectivity $_4$ such that $\pi 4(A'B'C'D_3) = A'B'C'D'$.

The resultant $\pi_4 \pi_3 \pi_2 \pi_I$ of these four collineations clearly transforms ABCD into A'B'C'D' respectively. If the quadrangles are in different planes, we need only add a perspective transformation between the two planes.

Corollary. There exists projective collineations in a plane which will effect any one of the possible twenty four permutations of the vertices of a complete quadrangle in the plane.

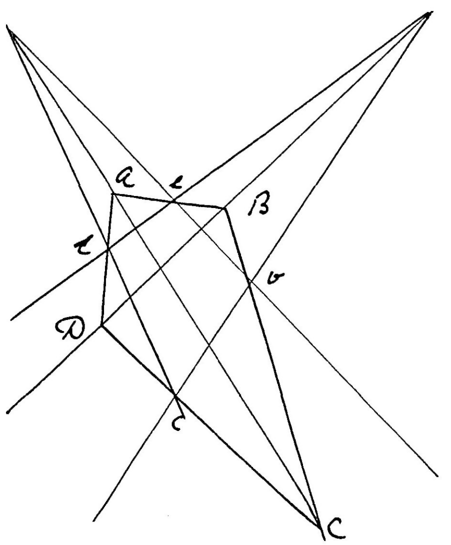
Theorem IV. The three circumferences of circles which have for their diameters the sides AC, BD and the diagonal line EF have, all three, the same point of intersection.



Proof: Suppose that the point which one has taken to draw the lines to the vertices and to the points of intersection of the opposite sides is one off the points of intersection of the two circles described on the sides AC and BD as diameters. The lines taken from this point to each pair will be perpendicular and it follows that the circumference described on EF as diameter will pass through C the same point of intersection of the circles on AC and BD.

Q.E.D.

Theorem V. If one of the four sides of the quadrangle ABCD one takes the four points a, b, c, d so that the relation $\frac{aA}{aB} \cdot \frac{bB}{bC} \cdot \frac{cC}{cD} \cdot \frac{dD}{dA} = I$ and if one considers these points as the consective vertices of a second quadrangle abcd the points of intersection of the opposite sides of this second quadrangle will be situated on the two sides AC, BD of the first.



The lines ab, cd will cross the line AC suppose in two distinct points e,e'. One will have therein the ABC, ADC the relation $\frac{aA}{aB} \cdot \frac{bB}{bC} \cdot \frac{eC}{eA} = I$

$\frac{cC}{cD} \cdot \frac{dD}{dA} \cdot \frac{e'A}{e'C} = I$

Multiply these together term for term,

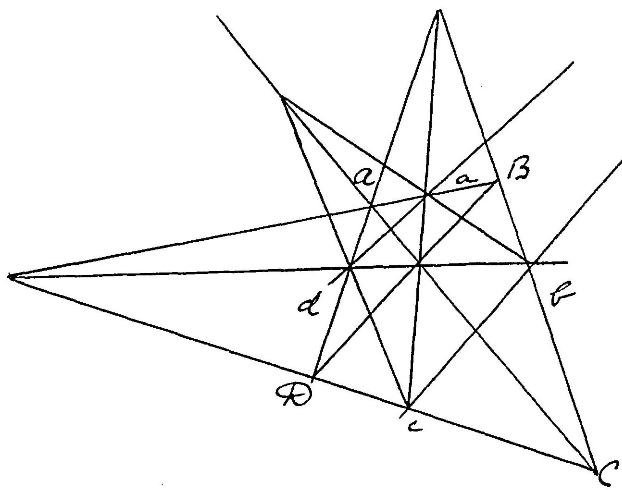
$$\frac{aA}{aB} \cdot \frac{bB}{bC} \cdot \frac{eC}{eA} \cdot \frac{cC}{cD} \cdot \frac{dD}{dA} \cdot \frac{e'A}{e'C} = I$$

or we must conclude $\frac{EC}{EA} = \frac{E'C}{E'A}$ or E andE' coincide.
Hence ab, cd, intersect on AC.

Q.E.D.

Theorem VI. If through the points of intersection of the opposite sides of a quadrangles ABCD one takes two lines which meet resepctively, the two pair of opposite sides in a, c, and b,d, there is formed between the segments which these points make on the four sides, the relation

$\frac{aA}{aB} \cdot \frac{bB}{bC} \cdot \frac{cC}{cD} \cdot \frac{dD}{dA} = I$ and it follows the quadrangle abcd which has for its vertices these four points has the points of intersection of the opposite sides on the sides AC and BD of the quadrangle.



The cross ratio of two ranges E,A,a,B andE,D,c,C are equal

$$\therefore \frac{aA}{aB} : \frac{EA}{eB} = \frac{cD}{cC} : \frac{eD}{EC}$$

and

$$\frac{bB}{bC} : \frac{FB}{FC} = \frac{dA}{dD} : \frac{FA}{FD}$$

These equations multiplied member by member gvies

$$\frac{aA}{aB} \cdot \frac{bB}{bC} \cdot \frac{cC}{cD} \cdot \frac{dD}{dA} = \frac{EA}{EB} \cdot \frac{EC}{ED} \cdot \frac{FA}{FB} \cdot \frac{FC}{FD}.$$

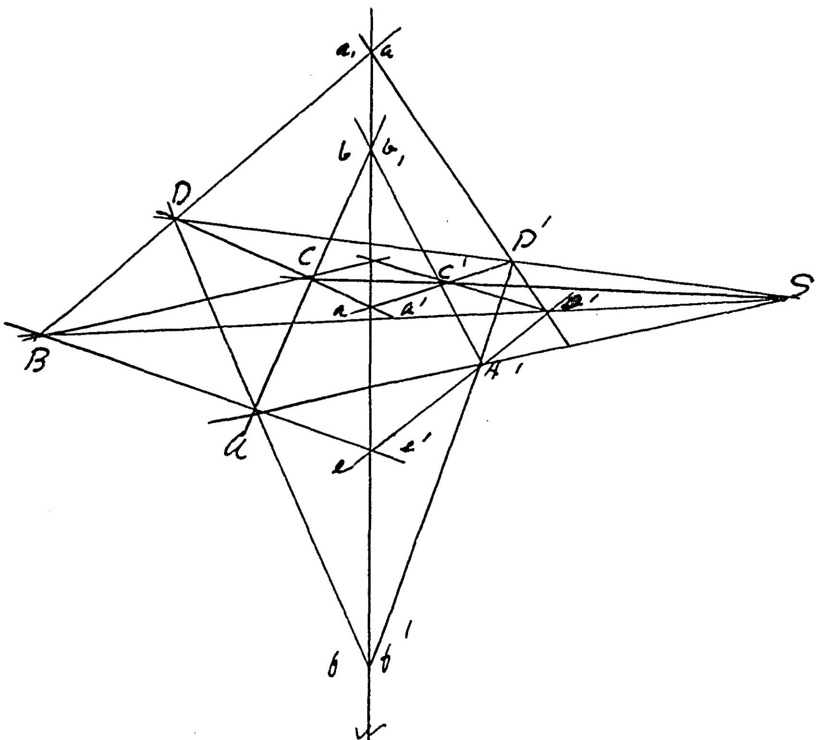
But the second member we know equals I.

$$\therefore \frac{aA}{aB} \cdot \frac{bB}{bC} \cdot \frac{cC}{cD} \cdot \frac{dD}{dA} = I$$

Q.E.D.

In connection with the complete quadrangle there are a number of theorems, which follow as special cases under some more general theorems and for which the author in whose book they may be found has thought it unnecessary to give a proof. Such properties are best understood by examining the previous work on which they are based, and for these only the theorem figures and the reference to the book in which they may be found, will be given. This same rule will hold also for those, where it is impossible to reproduce the proofs because of their length.

Theorem I. If two complete quadrangles ABCD and $A_I B_I C_I D_I$ lying in different planes whose line of intersection u passes through none of the eight vertices are correlated to each other, and five sides a, b, c, d, e, ' of one quadrangle intersect (upon u) the corresponding sides $a_I b_I c_I d_I e_I$ respectively of the other, there are the two quadrangles sections of one and the same complete focii-edge, and therefore their remaining two sides f and f_I also intersect on w.



If u is an infinitely distant straight line our theorem would read:

If, in two complete quadrangles which are correlated to each other five pair of homologous sides are parallel, then the remaining two sides are also parallel.

Geometry of Position by Reye.

Part I. Page 36- 37.

Theorem II. If two self polar quadangles ABCD and A B C'D'of a conic y^2 have two vertices A and B in common, their six vertices lie upon a curve of the second order, which may however degenerate into the two straight lines AB and CD.

Theorem . If a self polar triangle and a self polar quadrangle have one vertex in common, their six vertices lie upon a curve of the second order.

Theorem . If two self-polar quadrangles ABCD and A B'C'D' common to y^2 and y_I^2 , have one vertex A in common, their seven vertices lie upon a curve of the second order.

Geometry of Position by Reye.
Part I. Page 2I6.

Theorem III. If a conic section k^2 is circumscribed to a quadrangle which is self-polar with respect to another conic section y^2 the two curves have the following relation to each other:

(a) Any three points of k^2 determine a self polar quadrangle of y^2 whose fourth vertex also lies upon k^2.

(b) If k^2 is intersected by two straight lines which are conjugate with respect to y^2 the points of intersection are the vertices of a quadrangle which is self-polar with respect to y^2.

(c) To the curve k^2 there can therefore be inscribed an infinite number of quadrangles and are infinite of real triangles which are self-polar with respect to y^2; every point of k^2 enclosed by y^2 is a vertex of one of these self polar triangles.

(d) The polar of a point of k^2 with respect to y^2 intersects either one or both of the curves in two real points which are conjugate with respect to the curve upon which ~~the curve~~ they do not lie.

(e) Two tangents to y^2, whose points of contact are conjugate with respect to k^2, always intersect in a point of k^2.

Theorem IV. The conic section k^2 which supportca given curve of the second class y^2 form a mainfold of four dimensions which we shall call a net of conics of the fourth grade. About an arbitrary quadrangle there can be circumscribed in general only one conic of this net; for the quadrangle becomes self-polar with respect to y^2, and all conics circumscribed about it belong to the net, as soon as any of them support the curve y^2.

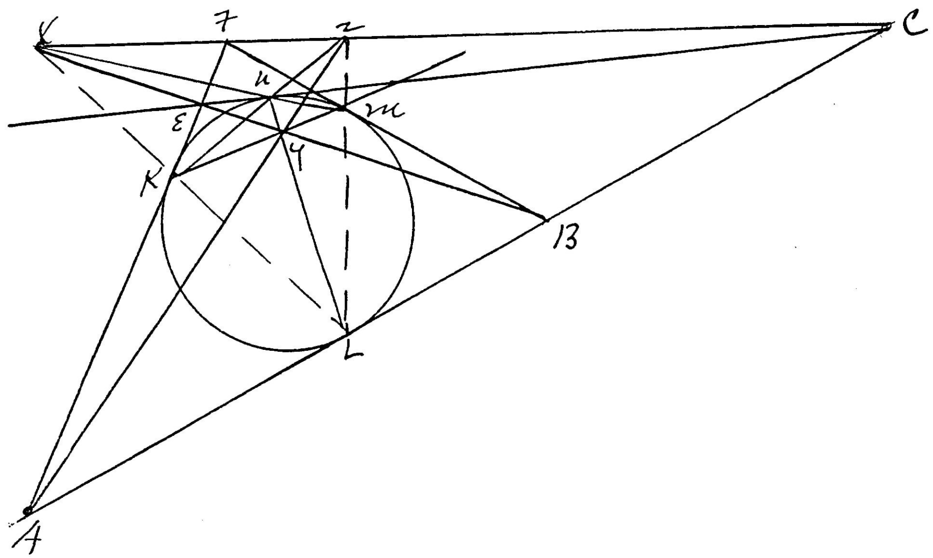
Theorem v. If two conics of a net of any grade are circumscribed to a quadrangle, this quadrangle is self-polar with respect to all curves of the second class which are supported by the net, and consequently all conics which circumscribe the quadrangle belong to the net.

Theorem . All conics which support two curves of the second class y^2 and y_I^2, and pass through two real points are circumscribed to a common self polar quadrangle of y^2 and y_I^2 whose remaining two vertices may however be conjugate imaginary.

Theorem vI. A point through which two conics of a sheaf pass is a comman point of all curves of the sheaf and a two fold point of the associate web of conics of the third grade. All the conics circumscribed to a quadrangle form a sheaf of conics; the associate web of the third grade contains all conics for which the quadrangle is self polar.

Theorem VII. A sheaf of conics contains at least one, but in general, and at most three real line pairs; these three pairs of rays intersect in the three vertices of the self-polar triangle at the sheaf, and form the three pairs of opposite sides of a real or imaginary quadrangle to which the sheaf of conics is circumscribed.

Geometry of Position by Reye
Part I Pages 217-299.



Theorem VIII. In any quadrangle inscribed in a curve of the second order, the point of intersection of pairs of opposite sides be in a straight line with the points of intersection of the tangents at opposite vertices.

Theorem IX. If four points K,L,M,N of a curve of the second order determine a complete quadrangle and then tangents k,l,m,n, a complete quadrilateral, the three pairs of opposite vertices of the quadrilateral lie upon the straight lines joining the points x,y,z, in which pairs of opposite sides of the quadriangle intersects.

Theroem X. If through a point N which lies in the plane of a curve of the second order, but not upon the curve, any number of secants of the curve be drawn and we determine-

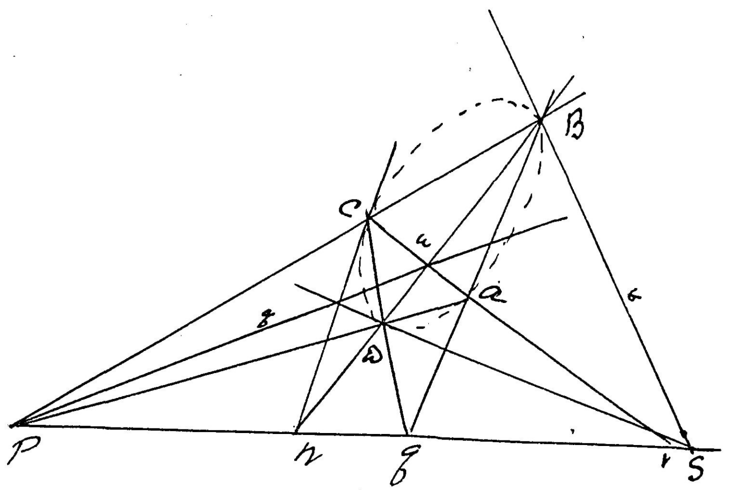
(I) The points of intersection of the pairs of opposite sides of any simple quadrangle inscribed in the curve which has two of these secants as diagonals;

(2) The point upon any secant which is harmonically separated from N by the points of intersection with the curve;

(3) The common point of the two tangents which can be drawn to the curve at the points of intersection of the secants;

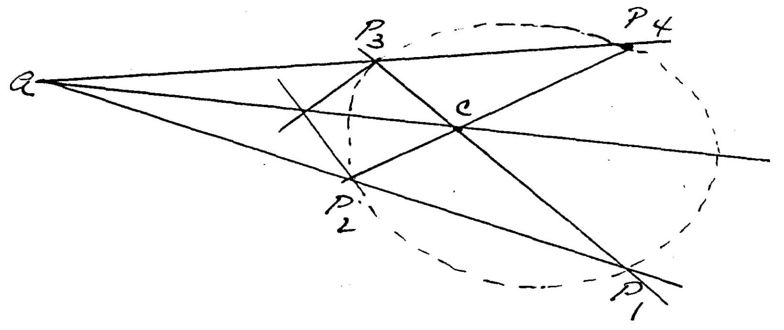
(4) The points of contact of the rangents which can be drawn to the curve from N;

Then are these points lie upon a straight line n which is caled the polar of the point with respect to the curve of the second order.



Geometry of Position by Reye. Page 100 of Patt I.
New York. 1898.

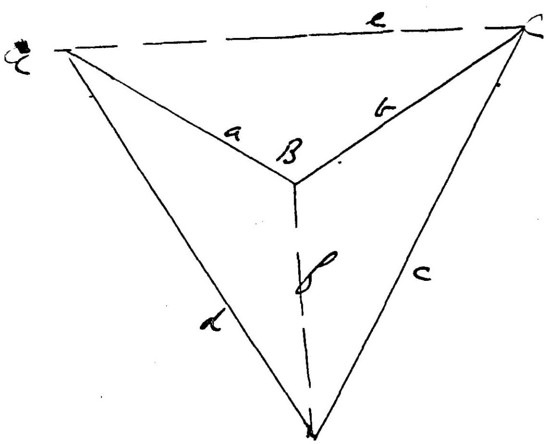
Theorem XI. If the vertices of a complete quadrangle are points of a point conic, the tangents at a pair of vertices meet in a point of the line passing through the diagonal points of the quadrangle which are not on the side joining the two vertices.

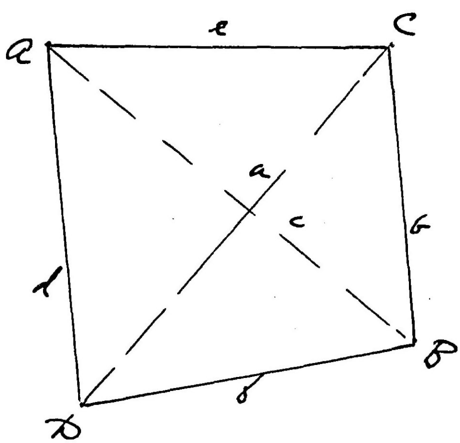


Taken from Volume I of Veblen and Young, Projective Geometry., page II5.

Theorem XII. In a complete quadrangle if $e:f = (da + bc) : (dc + ab)$ either $ef \quad ac \quad bd$ and the quadrangle may be inscribed or

$$\left\{ (a^2 + c^2 + b^2 + d^2 - e^2 - f^2)(ac + bd + ef) \right\} - 2(da + bc)(dc + ab) = 0$$





In case $ef = ac + bd$ and the quadrangle can not be inscribed the figure is called a Qyadrangle of Desboves. For a discussion of this one is referred to the articles by M.G. Fontene' in the January and February numbers of the Novelles Annales for ninteen hundred and eight.